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DIFFERENTIAL AND INTEGRAL CALCULUS

-lecture notes-

MATH6103 Differential & Integral Calculus
MATH6500 Elementary Mathematics for Engineers

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Chapter 1

Functions

1.1 What is a function?

Example 1.1. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the following equation,

$$A = \pi r^2. \tag{1.1}$$

With each positive number r, there is associated one value of A and we say that A is a function of r, denoted by $A = A(r)$.

Example 1.2. The human population of the world P depends on the time t. The table below gives estimates of the world population at time t, for certain years:

t (years)	1900	1910	1950	1960	1990	2000
P (millions)	1650	1750	2650	3040	5280	6080

For instance, $P(1950) \approx 2,560,000,000$ and $P(2000) \approx 6,080,000,000$.

For each value of time t, there is a corresponding value of P, and we say that P is a function of t.

However, in this case we don't know the rule that connects t and P at the moment. In fact, it is one of our tasks to find and understand the rule for a function.

1.2 The definition of a function

Here we will look at more precise definition of a function. First let us define a set.

Definition 1.1. *A* **set** *is a collection of objects, 'things' and states. A 'piece' or object of a set is called an element of the set.*

Example 1.3. The following are examples of sets:

(i) $\mathbb{Z} = \{\text{all whole numbers}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$,

(ii) $\mathbb{N} = \{\text{all positive whole numbers}\} = \{1, 2, 3, \dots\}$,

(iii) $\mathbb{R} = \{\text{all real numbers}\}$.

If we have two sets A and B, then a function $f : A \to B$ is a rule that sends each element x in A to exactly one element called $f(x)$ in B.

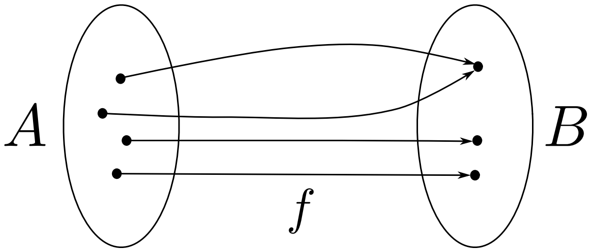

Figure 1.1: *Function f 'maps' elements from set A on to elements in set B. Many-to-one relationship.*

Here we call the set A the *domain* of f and B is known as the *range* of f. Suppose x is some element of A which is denoted by $x \in A$, then x is called an *independent variable* and $f(x)$ is called the dependent variable where $f(x) \in B$, i.e. $f(x)$ is an element of B.

NOTE: f is not a function if it is multi-valued, i.e. if one element from A maps to two distinct elements in B. Such a mapping is sometimes called multi-valued function, however this is just terminology, strictly by the definition for our purposes f would not be a function in this case.

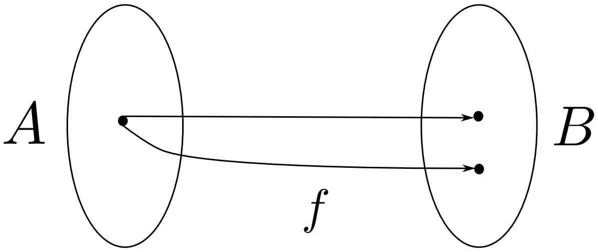

Figure 1.2: *The above 'map' f is not a function, i.e. when it is one-to-many or multi-valued.*

1.3 Representing a function

Example 1.4. Let S be the function from the real numbers to itself, that is $S : \mathbb{R} \to \mathbb{R}$ given by

$$S(x) = x^2. \tag{1.2}$$

Here we have defined the function S algebraically using an explicit formula.

That is, S is the function that squares things (real numbers), i.e. a function performs some sort of ‘action’. Here we have described the function verbally!

We can also represent a function numerically (by table of values):

x	1	2	3	...	7	...
$S(x)$	1	4	9	...	49	...

Finally, a function can also be represented visually, for example by a graph.

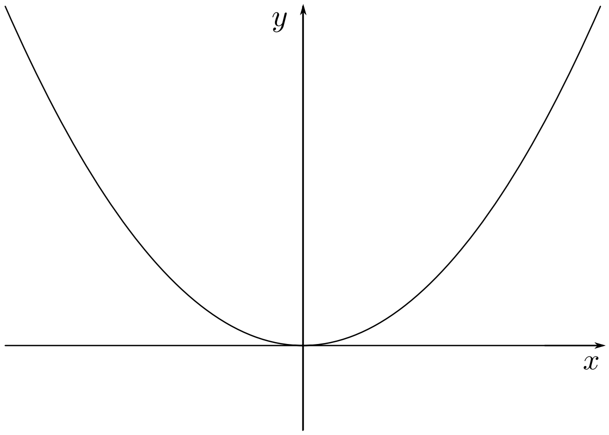

Figure 1.3: *Graph displaying the function* $y = S(x) = x^2$.

If the domain of S is $\mathbb{R}$, then the range of S is $\mathbb{R}^+ = \{\text{all positive real numbers}\}$.

Summary:

A function is a rule, it takes some independent variable (in the domain) for which there is some dependent variable (in the range).

There are four ways to represent function.

In the following, we take a look at some specific functions.[1]

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[1] End Lecture 1.

1.3.1 Exponents

We have seen the function $f(x) = x^2$. In general, a function of the form $f(x) = x^a$, where a is a constant, is called a power function. For example,

$$\begin{gathered} f(x) = x^5, \quad a = 5, \\ f(x) = \sqrt{x}, \quad a = 1/2, \quad x \geq 0. \end{gathered}$$

What about $f(x) = a^x$? Can we define a function of this form? Yes, we can, but we need to explore the meaning of a^x, when x is not a positive integer.

REVISION: when we wish to multiply a number by itself several times, we make use of index or power notation. We have notation for powers:

$$a^2 = a \cdot a, \quad a^3 = a \cdot a \cdot a, \quad a^x = \overbrace{a \cdot a \ldots a \cdot a}^{x}, \quad a \in \mathbb{R}, \quad x \in \mathbb{N}.$$

Here, a is called the *base* and x is called the *index* or *power*. We also know the following properties (laws of exponents)

1. $\boxed{a^{x+y} = a^x \cdot a^y}$, e.g. $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^3 \cdot 2^2, \quad x = 3, \quad y = 2.$
2. $\boxed{(a^x)^y = a^{xy}}$, e.g. $3^6 = 3^{2 \times 3} = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^2 \cdot 3^2 \cdot 3^2 = (3^2)^3.$
3. $\boxed{a^x \cdot b^x = (ab)^x}$, e.g. $2^3 \cdot 3^3 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = (2 \cdot 3)(2 \cdot 3)(2 \cdot 3) = (2 \cdot 3)^3.$

These laws hold for any $a \in \mathbb{R}$ and $x, y \in \mathbb{N}$.

Now we want to generalise the notation for the power to include whole numbers, fractions and irrational numbers, so that a^x makes sense for most values of a and all $x \in \mathbb{R}$. The idea is to make laws of exponents hold generally.

First, we choose $a^0 = 1$ ($a \neq 0$) so that

$$a^2 = a^{2+0} = a^2 \cdot a^0 = a^2, \quad \text{i.e. } a^x = a^{x+0} = a^x \cdot a^0 = a^x.$$

Second, we choose

$$a^{-1} = \frac{1}{a}, \quad a^{-2} = \frac{1}{a^2}, \ldots, a^{-n} = \frac{1}{a^n}, \quad n \in \mathbb{N},$$

so that

$$a^2 \cdot a^{-2} = a^{2-2} = a^0 = 1, \quad \text{or } a^n \cdot a^{-n} = a^{n-n} = a^0 = 1, \quad \text{for all } n \in \mathbb{N}.$$

We choose

$$a^{1/2} = \sqrt{a} \quad (\text{the square root of } a,\ a \geq 0),$$

$$\begin{aligned}
&a^{1/3} = \sqrt[3]{a} \quad \text{(the cube root of } a,\ a \in \mathbb{R}\text{)},\\
&\vdots\\
&a^{1/n} = \sqrt[n]{a} \quad \text{(the } n\text{th root of } a\text{. If } n \text{ is even, } a \geq 0\text{; otherwise any } a \in \mathbb{R} \text{ is O.K.)}
\end{aligned}$$

so that

$$\begin{aligned}
&a^{1/2} \cdot a^{1/2} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a,\\
&a^{1/3} \cdot a^{1/3} \cdot a^{1/3} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a,\\
&\vdots\\
&\overbrace{a^{1/n} \cdot a^{1/n} \ldots a^{1/n}}^{n} = a^{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}} = a^{\frac{n}{n}} = a^1 = a.
\end{aligned}$$

If x is a rational number, then $x = p/q$, where p and q are integers and $q > 0$. Then

$$a^x = a^{p/q} = (a^{1/q})^p = (a^p)^{1/q} \quad \text{(order doesn't matter)}.$$

Therefore, a^x makes sense for any rational number x.

If x is an irrational number, then we can always find two rational numbers c and d which are sufficiently close to x and which satisfy $c < x < d$. So

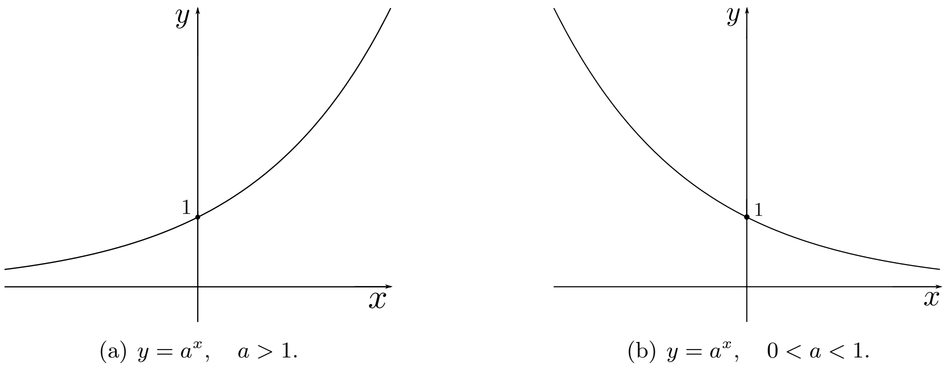
$$a^c < a^d \quad \text{if} \quad a \geq 1; \quad a^c > a^d \quad \text{if} \quad 0 < a < 1.$$

It can be shown that there is exactly one number between a^c and a^d. We define this number as a^x.

Finally, an exponential function can be defined by

$$f(x) = a^x, \quad x \in \mathbb{R},$$

where a is a positive constant. The domain of f is $\mathbb{R}$ and the range is $\mathbb{R}^+$. Graph:



(a) $y = a^x, \quad a > 1.$ (b) $y = a^x, \quad 0 < a < 1.$

Figure 1.4: *Comparison of exponent graphs for different values of a.*

Summary:

If a and b are positive numbers, x and y are any real numbers, then we have

1 $a^{x+y} = a^x \cdot a^y$,

2 $a^{x-y} = \frac{a^x}{a^y}$,

3 $(a^x)^y = a^{xy}$,

4 $a^x \cdot b^x = (ab)^x$.

There is also a special exponential function, $f = e^x$, we will investigate this further later in the course.[2]

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[2] End Lecture 2.

1.3.2 Polynomials

A polynomial is a function P with a general form

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \tag{1.4}$$

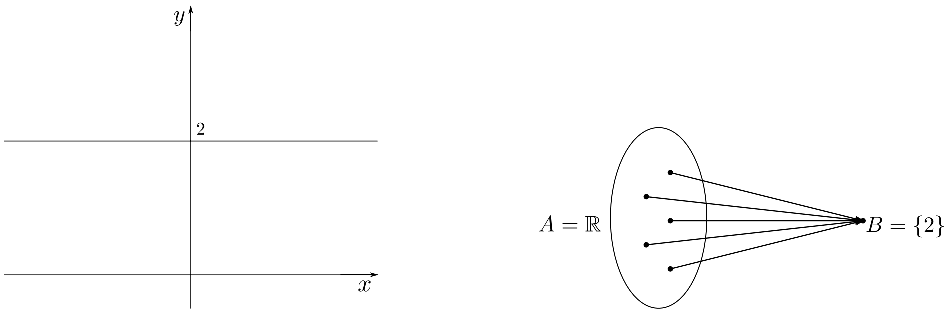
where the coefficients a_i $(i = 0, 1, \ldots, n)$ are numbers and n is a non-negative whole number. The highest power whose coefficient is not zero is called the *degree* of the polynomial P.

Example 1.5.

$P(x)$	2	$3x^2+4x+2$	$\frac{1}{1+x}$	$\sqrt{x}$	$1-3x+\pi x^3$	$2t+4$
Polynomial?	Yes	Yes	No	No	Yes	Yes
Order	0	2	N/A	N/A	3	1

Importance of polynomials: analytical & computational points of view

Degree 0: $P(x) = a_0 = a_0x^0$, say $P(x) = 2$. This polynomial is simply a constant.



(a) $y = P(x) = 2$.

(b) *Entire domain mapped to one point.*

Figure 1.5: *A straight line parallel to the x-axis.*

Degre 1: $P(x) = ax + b$, $a \neq 0$. These are called *linear*, since the graph of $y = ax + b$ is a straight line. The linear equation $ax + b = 0$ has solution $x = -b/a$.

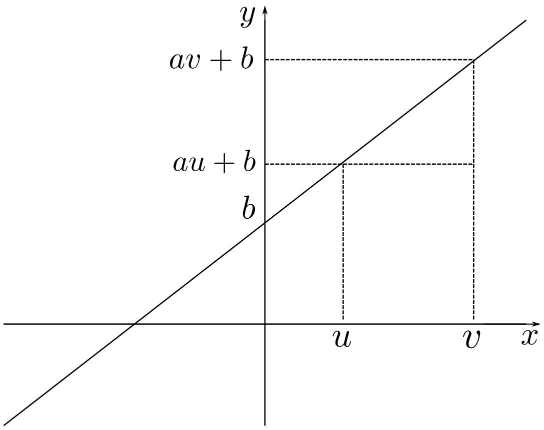


Figure 1.6: *Linear graph given by $y = ax + b$.*

The *slope* or *gradient* of $y = ax + b$ is a. It can be worked out as follows:

$$\begin{aligned}
\textit{slope} &= \frac{\text{change in height}}{\text{change in distance}} \\
&= \frac{\text{change in } y}{\text{change in } x} \\
&= \frac{(av+b)-(au+b)}{v-u} \\
&= \frac{a(v-u)}{v-u} = a. \qquad (1.5)
\end{aligned}$$

Degree 2: $P(x) = ax^2 + bx + c$, $a \neq 0$. This is known as a quadratic polynomial. The quadratic equation $ax^2 + bx + c = 0$ has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (1.6)$$

Proof. Start by re-arranging the equation and dividing through by a so that

$$x^2 + \frac{b}{a}x = -\frac{c}{a},$$

next we add $b^2/(2a)^2$ to both sides, hence

$$x^2 + \frac{b}{a}x + \frac{b^2}{(2a)^2} = -\frac{c}{a} + \frac{b^2}{(2a)^2}.$$

Now we can search for a common denominator on the right hand side (RHS) and complete the square or factorise on the left hand side (LHS), i.e.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{2^2ac}{(2a)^2} + \frac{b^2}{(2a)^2}.$$

Taking the square root of both sides we have

$$x + \frac{b}{2a} = \pm\sqrt{-\frac{2^2ac}{(2a)^2} + \frac{b^2}{(2a)^2}} = \frac{\pm\sqrt{-2^2ac + b^2}}{2a},$$

and finally re-arranging the above equation (or minus $b/2a$ from both sides of the equation) we get[3]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$\square$

Example 1.6. Consider the quadratic polynomial

$$P(x) = x^2 - 3x + 2. \tag{1.7}$$

We can represent $P(x)$ in a different way by factorising it, i.e.

$$P(x) = (x - 2)(x - 1). \tag{1.8}$$

Again, we can represent $P(x)$ in a different way, this time by completing the square. When completing the square, we do this based on the value of b as follows. We add and subtract $\left(\frac{b}{2}\right)^2$ to $P(x)$ such that $P(x) = x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2$ (i.e. we don't really change the equation). Now it is easy to see $x^2 - 3x + \left(\frac{3}{2}\right)^2$ is the same as $\left(x - \frac{3}{2}\right)^2$ i.e. it can be factorised. Thus we can finally write

$$P(x) = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}. \tag{1.9}$$

Now, (1.7), (1.8) and (1.9) are all equivalent and they are each able to give an insight on what the graph of the quadratic function $P(x)$ looks like. That is

(i) Equation (1.7) tells us the graph of $P(x)$ is a "cup" rather than a "cap" since the coefficient of x^2 is positive. Also we can easily see $P(0) = 2$.

(ii) Equation (1.8) tells us $P(x) = 0$ at $x = 1$ and $x = 2$

(iii) Equation (1.9) tells us $P(x)$ is minimal at $x = \frac{3}{2}$ and $P\left(\frac{3}{2}\right) = -\frac{1}{4}$. Note,

$$\left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4},$$

since anything squared is always positive!

Now that we have some suitable information regarding $P(x)$, we are able to produce an informed sketch:

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[3] Aside: $\square \equiv Q.E.D$, where $Q.E.D =$ *"quod erat demonstratum"* which means *"which was to be demonstrated"* in Latin.

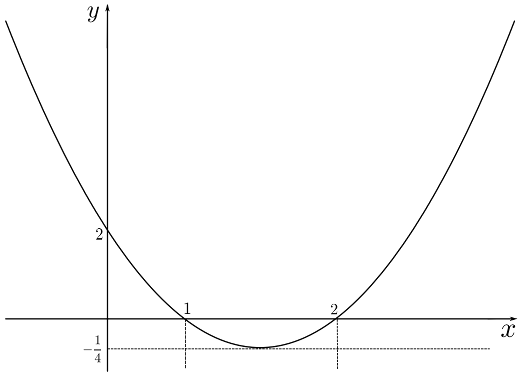

Figure 1.7: *Quadratic graph given by* $y = x^2 - 3x + 2$.

Degree ≥ 3: Things get a bit more complicated!

In general, we have the algebraic equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0, \tag{1.10}$$

which has n roots, including real and complex (imaginary numbers, $z = \alpha + i\beta$) roots.

$n = 2$	we have formulae for roots	(quadratics)
$n = 3$	we have formulae for roots	(cubic)
$n = 4$	we have formulae for roots	(quartics)
$n > 4$	No general formulae exist	(proven by Évariste Galois)

But in any case, we may try factorisation to find the roots. If you factorise a polynomial, say

$$P(x) = (x-1)(x+3)(x+4),$$

then you can easily solve $P(x) = 0$, in this case $x_1 = 1$, $x_2 = -3$, $x_3 = -4$.

NOTE: this depends on the property of 0 on the RHS. You can't easily solve

$$(x-1)(x+3)(x+4) = 1.$$

Conversely, if you know that $P(\alpha) = 0$, then you may factorise $P(x)$ as

$$P(x) = (x - \alpha)q(x),$$

where $q(x)$ is some polynomial of one degree less than that of $P(x)$.

Example 1.7. Consider $P(x) = x^3 - 8x^2 + 19x - 12$. We know that $x = 1$ is a solution to $P(x) = 0$, then it can be shown that

$$P(x) = (x-1)q(x) = (x-1)(x^2 - 7x + 12).$$

Here $P(x)$ is a cubic and thus $q(x)$ is a quadratic.

Example 1.8. Consider $P(x) = x^3 - x^2 - 3x - 1$. By observation, we know

$$P(-1) = (-1)^3 - (-1)^2 - 3(-1) - 1 = 0.$$

So $x_1 = -1$ is a root. Let us write

$$P(x) = (x+1)(x^2 + ax + b),$$

then multiplying the brackets we have

$$P(x) = x^3 + (1+a)x^2 + (a+b)x + b,$$

which should be equivalent to $x^3 - x^2 - 3x - 1$. Thus, comparing the corresponding coefficients we have

$$\begin{aligned} 1 + a &= -1, \\ a + b &= -3, \\ b &= -1. \end{aligned}$$

This set of simultaneous equations has the solution

$$a = -2, \quad b = -1.$$

So we can write

$$P(x) = (x+1)(x^2 - 2x - 1)$$

To find the other two solutions of $P(x) = 0$, we must set $(x^2 - 2x - 1) = 0$ which has solution $x_{2,3} = \frac{2\pm\sqrt{8}}{2} = 1 \pm \sqrt{2}$, together with $x_1 = -1$ we have a complete set of solutions for $P(x) = 0$.

Example 1.9. Consider $P(x) = x^3 + 3x^2 - 2x - 2$. An obvious solution to $P(x) = 0$ is $x_1 = 1$. So we put

$$\begin{aligned} P(x) &= (x-1)(x^2 + ax + b) \\ &= x^3 + (a-1)x^2 + (b-a)x - b. \end{aligned}$$

Comparing corresponding coefficients with our original form of $P(x)$ we gain the following simultaneous equations:

$$\begin{aligned} a - 1 &= 3, \\ b - a &= -2, \\ -b &= -2. \end{aligned}$$

These have solution $b = 2$, $a = 4$ and so we have

$$x^3 + 3x^2 - 2x - 2 = (x-1)(x^2 + 4x + 2).$$

The solutions to $(x^2 + 4x + 2) = 0$ are $x_{2,3} = -2 \pm \sqrt{2}$, completing the set solutions to $P(x) = 0$.

NOTE: In the above examples, the leading coefficient of $P(x)$ i.e. the coefficient of x^3 is equal to one!

As with many areas of mathematics, there are many ways to tackle a problem. Another way to find $q(x)$ given you know some factor of $P(x)$, is called *polynomial devision.*

Example 1.10. Consider $P(x) = x^3 - x^2 - 3x - 1$, we know $P(-1) = 0$. The idea is that we “divide” $P(x)$ by the factor $(x + 1)$, like so:

$$\begin{array}{rrrrr}
 & & x^2 & -2x & -1 \\ \cline{2-5}
x+1\big) & x^3 & -x^2 & -3x & -1 \\
 & -x^3 & -x^2 & & \\ \cline{2-3}
 & & -2x^2 & -3x & \\
 & & 2x^2 & +2x & \\ \cline{3-4}
 & & & -x & -1 \\
 & & & x & +1 \\ \cline{4-5}
 & & & & 0
\end{array}$$

Hence, multiplying the *quotient* by the *divisor* we have $(x+1)(x^2-2x-1) = x^3-x^2-3x-1$.

Summary:

The highest power of the polynomial is known as the order of the polynomial.

The roots of a quadratic equation $ax^2 + bx + c$, can be found using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

No general formula for $n > 4$. However roots can be found of higher order polynomials by factorising first. This can be done by observation, the method of comparing coefficients or by long polynomial devision.

Later we will see that polynomials are functions which are easy to differentiate and integrate.

We can approximate most “nice” functions by polynomials, at least locally i.e. using power series.[4]

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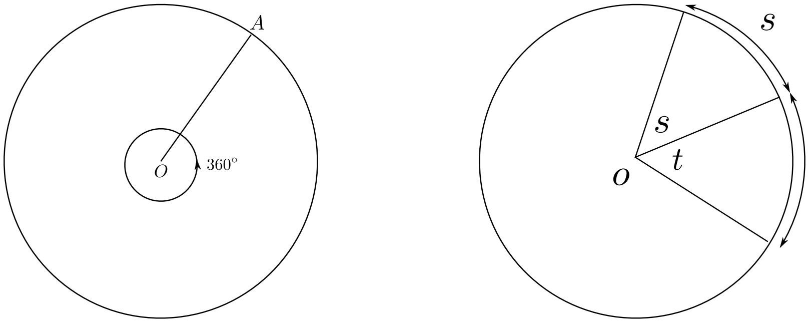
[4] End Lecture 3.

1.3.3 Trigonometric functions

The measure of an angle

Degrees: One full turn is 360°.

Radians: (Usually abbreviated as "rad" or the superscript " ^{c} " meaning "circular measure"). Suppose we have a disc of radius 1, we choose a couple of radii and want to measure the angle between them. To do so, we measure the length of the arc between the radii. Suppose the length is t. This is what we call the size of the angle, i.e. it is another measure for the length of the arc.



(a) 360° *is a full turn.* (b) *Addition of angles.*

Figure 1.8: *Some properties of angles.*

We can see immediately that a full turn is 2π rad because a circle of radius 1 has a circumference 2π. Therefore we have

$$\begin{aligned} 1 \text{ turn} &= 360^\circ = 2\pi \,\text{rad}, \\ \frac{1}{2} \text{ turn} &= 180^\circ = \pi \,\text{rad}. \end{aligned} \tag{1.11}$$

So,

$$\begin{aligned} 1 \,\text{rad} &= \frac{180^\circ}{\pi}, \\ 1^\circ &= \frac{\pi}{180} \,\text{rad}. \end{aligned} \tag{1.12}$$

If the radius is not 1, then you need to take the ratio

$$\frac{\text{arc length}}{\text{radius}} = \text{angle (in radians)}. \tag{1.13}$$

Radians and degrees have one important property in common: if you follow one angle by another then the total angle is the sum e.g. $t + s$ (see Fig. 1.8(b)). This means that addition of angles has a geometric meaning.

Trignometric functions: cosine, sine & tangent

The values $\cos(\theta)$ and $\sin(\theta)$ (usually written $\cos\theta$, $\sin\theta$) are the horizontal and vertical coordinates of the point C (see Fig. 1.9).

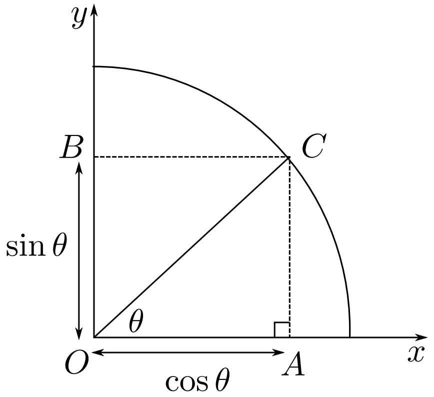


Figure 1.9: *Geometric definition of* $\cos$ *and* $\sin$*, circle radius* $r = OC = 1$.

$$\cos\theta = \frac{OA}{OC} = OA, \tag{1.14a}$$

$$\sin\theta = \frac{AC}{OC} = AC, \tag{1.14b}$$

$$\tan\theta = \frac{AC}{OA} = \frac{\sin\theta}{\cos\theta}, \quad (\cos\theta \neq 0). \tag{1.14c}$$

A number of properties are clear:

1. $\cos^2\theta + \sin^2\theta = 1$

   $$\cos^2\theta + \sin^2\theta = \left(\frac{AC}{OC}\right)^2 + \left(\frac{BC}{OC}\right)^2 = \frac{AC^2 + BC^2}{OC^2} = \frac{OC^2}{OC^2} = 1.$$

   Note: $AC^2 + BC^2 = OC^2$ by Pythagoras' theorem.

2. cos and sin are periodic functions with period 2π, i.e. for any x, $\cos(x + 2\pi) = \cos x$, $\sin(x + 2\pi) = \sin x$.

   In general, if $f(x + T) = f(x)$ for all x, then we say that $f(x)$ is periodic function with period T.

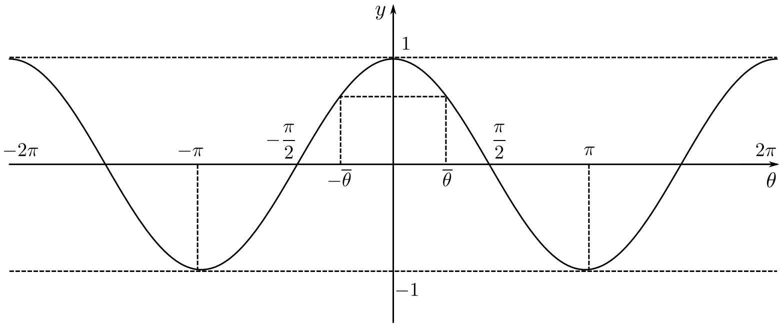


Figure 1.10: *Graph of* $\cos\theta$.

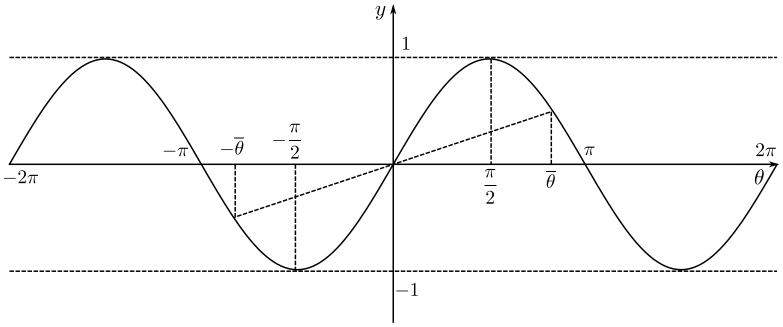


Figure 1.11: *Graph of* $\sin\theta$.

Note: $\cos : \mathbb{R} \to [-1,1]$ and $\sin : \mathbb{R} \to [-1,1]$.

3. cos is even, i.e. $\cos(-\theta) = \cos\theta$. sin is odd, i.e. $\sin(-\theta) = -\sin\theta$.

   In general, if $f(-x) = f(x)$ for all x, then we say f is even. If $f(-x) = -f(x)$ for all x, then we say f is odd.

4. Shift: cos and sin are the same shape but shifted by $\pi/2$, which means

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \quad \text{or} \quad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta,$$

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta, \quad \text{or} \quad \sin\left(\theta - \frac{\pi}{2}\right) = \cos\theta.$$

5. Addition formulae:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

6. Double-angle formulae:

$$\begin{aligned}\cos(2\theta) &= \cos(\theta+\theta)\\ &= \cos^2\theta - \sin^2\theta\\ &= \cos^2\theta - (1-\cos^2\theta)\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta.\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= \sin(\theta+\theta)\\ &= 2\sin\theta\cos\theta.\end{aligned}$$

7. Half-angle formulae: Let $2\theta = \alpha$, then

$$\begin{aligned}\cos\alpha = 2\cos^2\left(\frac{\alpha}{2}\right) - 1 \quad &\Longrightarrow \quad \cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}\\ \cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right) \quad &\Longrightarrow \quad \sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}\end{aligned}$$

Exercise 1.1. Try find $\cos(\theta-\phi)$ and $\sin(\theta-\phi)$ using properties (3) and (5).

We define tangent (tan) as follows:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cos\theta \neq 0.$$

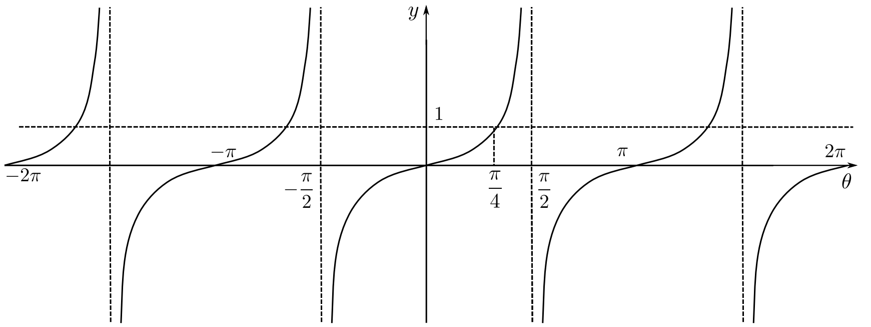


Figure 1.12: *Graph of* $\tan\theta$.

NOTE: $\tan : \mathbb{R} \to \mathbb{R}$. Also, as $\theta \to \pm\frac{\pi}{2}(2N-1)$, $\tan\theta \to \infty$ for $N \in \mathbb{N}$, this is the same as saying, as $\cos\theta \to 0$, $\tan\theta \to \infty$. We say the function has vertical *asymptotes* at these points. Also note that $\tan\theta$ has a periodicity of π!

The double angle formula for tan can be calculated using the definition as follows:

$$\begin{aligned}\tan(\theta+\phi) &= \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)}\\ &= \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi},\end{aligned}$$

dividing by $\cos\theta\cos\phi$ we get

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}.$$

Also, if we divide property (1) on page 14 through by $\cos^2 x$, we have

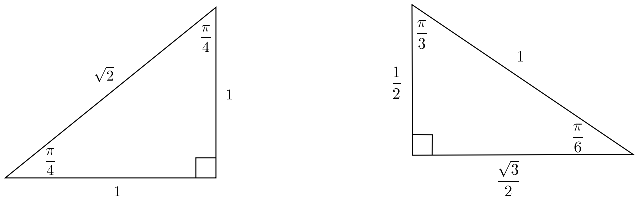
$$1 + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x},$$

which can be written as

$$1 + \tan^2 x = \sec^2 x.$$

The secant, cosecant and cotangent functions are defined as

$$\sec x = \frac{1}{\cos x}, \quad \mathrm{cosec} x = \frac{1}{\sin x}, \quad \mathrm{cot} x = \frac{1}{\tan x}.$$



(a) $\pi/4$ *triangle.* (b) $\pi/3$, $\pi/6$ *triangle.*

Figure 1.13: *Some well known results for particular angles can be derived by the above triangles for* sin, cos *and* tan.

Summary:

1 radian = $180/\pi$, and 1 degree = $\pi/180$.

The ratio of arc length to radius gives the angle (in radians)

$\cos\theta$ and $\sin\theta$ have a periodicity of 2π, whilst $\tan\theta$ has a periodicity of π (convince yourself why!).

The range of $\cos\theta$ and $\sin\theta$ is $[-1, 1]$, whilst the range of $\tan\theta$ is $(-\infty, \infty)$.

$\sin\theta$ and $\cos\theta$ can be obtained by shifting one another by $\pi/2$.

Take note of the various angle formulae![5]

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[5] End Lecture 4.

Chapter 2

Differentiation

2.1 Rates of change

Suppose we drive from London to Birmingham (100 miles). We plot a graph of the distance traveled against elapsed time. We want to measure how fast we traveled. The average speed of the trip is calculated as follows:

$$\frac{100 \text{ miles}}{2 \text{ hrs}} = 50 \text{ mph}.$$

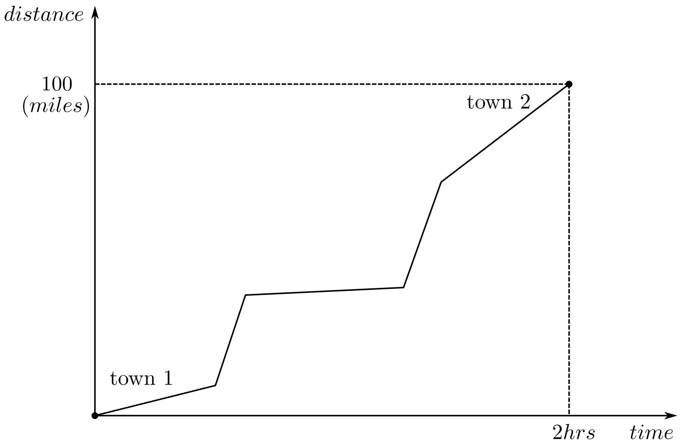

Figure 2.1: *Graph showing distance travelled against time, from town 1 to town 2.*

However, when travelling on the motorway you do not stick to one speed, sometimes you do more than 50 mph, sometimes much less. The reading on your speedometer is your *instantaneous* speed. This corresponds to the *gradient* (slope) “at a point” of the graph given in Fig. 2.1.

In the following, we want to find the slopes of curves.

2.1.1 What is the slope of a curve at a point?

The slope of a curve at a point is the slope of the *tangent* line at that point.

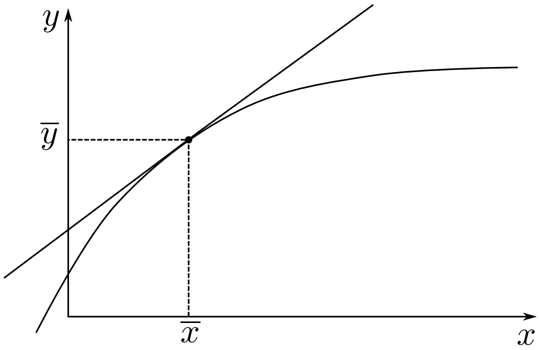

Figure 2.2: *Curve* $y = f(x)$ *with tangent line at* $(\overline{x}, \overline{y})$.

A straight line touching the edge of a curve (but not cutting across it) is called the tangent line at the touching point on the curve.

We hope that for mathematical curves, we find slopes “algebraically”.

Example 2.1. Let us start with the example of the curve $y = S(x) = x^2$.

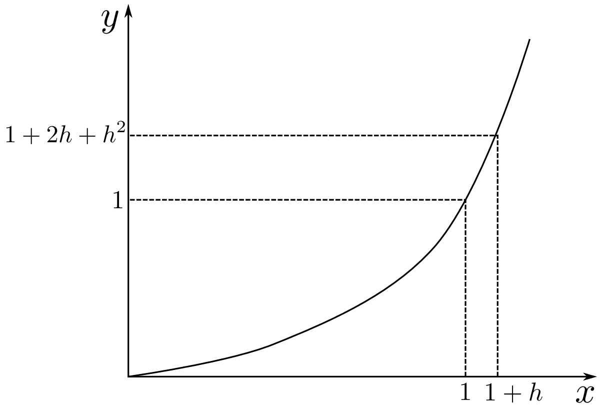

Figure 2.3: *Curve* $y = S(x) = x^2$ *displaying small increment about* $x = 1$.

Look at the point $(1, 1)$ on the curve. We want to understand how this function behaves when x is close to 1. We therefore choose some x close to 1, i.e. $x = 1 + h$ where $h \ll 1$. So we have

$$S(1 + h) = (1 + h)^2 = 1 + 2h + h^2 = S(1) + 2h + h^2.$$

This tells us two things:

1. Near 1, the value of $S(1)$ is also near 1, since if h is small, then $2h+h^2$ is also small. This is the concept of continuity, which says:
$$f(x_0) \to f(x) \quad \text{as} \quad x_0 \to x,$$
for a continuous function f.
2. If you increase x from 1 to $1+h$, then you increase the value of S from $S(1)$ to $S(1)+2h+h^2$, which is an increase of approximately $2h$, since if h is small then h^2 is very, very small. (e.g. $h = 0.001$, $h^2 = 0.00001$).

From (2), we know that as you increase the value of x from 1 to $1+h$, S in creases by approximately twice as much as h. Therefore the rate of change at 1, which is defined as the derivative of the function $S(x) = x^2$, is $S'(1) = 2$.

What about when $x = c$? As x changes from c to $c+h$, $S(x) = x^2$ changes from c^2 to $(c+h)^2 = c^2 + 2ch + h^2 = S(c) + 2ch + h^2$. The change is roughly $2ch$. That is, x has changed by h and x^2 has changed by $2c$ times as much. So $S'(c) = 2c$. We say the derivative of S at c is $2c$. In general,
$$S'(x) = 2x.$$

So, we have the idea that the derivative of a function f at $x = c$ is the coefficient of h in the expansion of $f(c+h)$.

Example 2.2. Let us consider the function $q(x) = x^3$. At $x = c+h$ we have
$$q(c+h) = (c+h)^3 = c^3 + 3c^2h + 3ch^2 + h^3.$$
Therefore, given what we have learnt, we can say
$$q'(c) = 3c^2 \quad \text{i.e.} \quad q'(x) = 3x^2.$$

PROBLEM: Suppose we want to examine the function $r(x) = 1/x$. How do we expand $r(c+h) = 1/(c+h)$?

Before we tackle this problem, let us turn back to Ex. 2.2. First let us write $q(c) = c^3$. Now, to get the change from $q(c+h)$ we have
$$q(c+h) - q(c) = 3c^2h + 3ch^2 + h^3.$$
We now look at the ratio
$$\frac{\text{change in } q}{\text{change in } x} = \frac{q(c+h)-q(c)}{(c+h)-c} = \frac{3c^2h + 3ch^2 + h^3}{h} = 3c^2 + 3ch + h^2.$$
We now examine what happens as h approaches zero. In this case
$$\lim_{h\to 0} \left\{3c^2 + 3ch + h^2\right\} = 3c^2.$$

Geometrically, we calculate the slope of the chord joining points on the curve (e.g. $q(c)$ to $q(c+h)$).

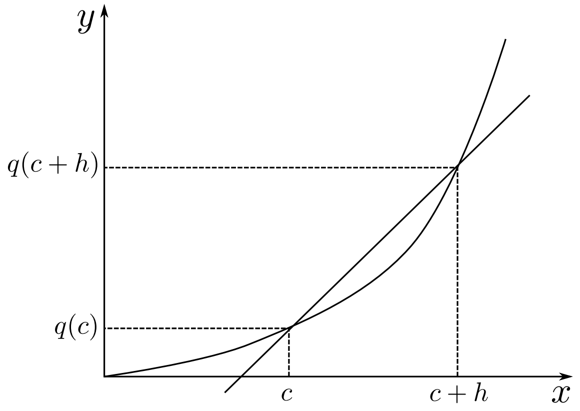

Figure 2.4: *Graph showing chord (line) joining the points $(c, q(c))$ and $(c+h, q(c+h))$ on the curve $y = q(x)$.*

The slope of the chord is

$$\frac{q(c+h)-q(c)}{h}, \tag{2.1}$$

and we watch what happens to the slope of the chord as $c+h$ gets closer and closer to c (i.e. h gets smaller and smaller) in the hope that the slope of the chord will approach the slope of the tangent line!

Example 2.3. Now let us consider the function $r(x) = 1/x$. In this case we have

$$r(c+h) - r(c) = \frac{1}{c+h} - \frac{1}{c}.$$

Now, let us consider the ratio

$$\frac{r(c+h)-r(c)}{h} = \frac{1}{h}\left(\frac{1}{c+h} - \frac{1}{c}\right) = \frac{1}{h}\left(\frac{-h}{c(c+h)}\right) = -\frac{1}{c(c+h)},$$

and as $h \to 0$, we have

$$r'(c) = -\frac{1}{c^2}, \quad \text{i.e.} \quad r'(x) = -\frac{1}{x^2}, \quad x \neq 0.$$

NOTE: $r(x) = 1/x$ is not well defined at $x = 0$ and in this case, nor is its derivative.[1]

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[1] End Lecture 5.

Definition 2.1. *In general, we define the derivative of a function f at x as*

$$f'(x) = \frac{d}{dx}\left(f(x)\right) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \tag{2.2}$$

provided that the limit exists. If the limit exists, we say f is differentiable at x. If we simply say f is differentiable, we mean f is differentiable at all values of x. In this case, $f'(x)$ is also a function of x.

Comments:

1. We interpret the derivative as the instantaneous rate of change, or geometrically as the slope of the tangent line.
2. Equivalently,

   $$f'(x) = \lim_{x_0\to x} \frac{f(x_0)-f(x)}{x_0-x}, \tag{2.3}$$

   since if you put $h = x_0 - x$, then $h \to 0 \iff x_o \to x$ and $f(x_0) = f(x+h)$.

Example 2.4. An example of a function which is not differentiable at a certain point:

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} .$$

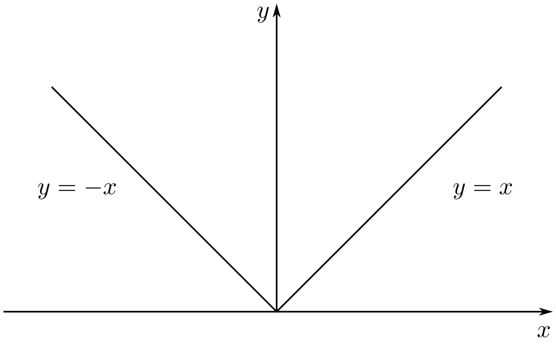


Figure 2.5: *Graph of $y = |x|$.*

At $x = 0$, $f(x)$ is continuous but not differentiable, since through the point $(0,0)$, you can draw many, many tangent lines. We can also show

$$\text{for} \quad h > 0, \quad \frac{f(0+h)-f(0)}{h} = \frac{h-0}{h} = 1,$$

$$\text{for} \quad h < 0, \quad \frac{f(0+h)-f(0)}{h} = \frac{-h-0}{h} = -1,$$

i.e.

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

doesn't exist! Taking the limit from both sides must give the same answer.

Example 2.5. What is the derivative of x^n for a positive whole number n? Let $f(x)=x^n$, then

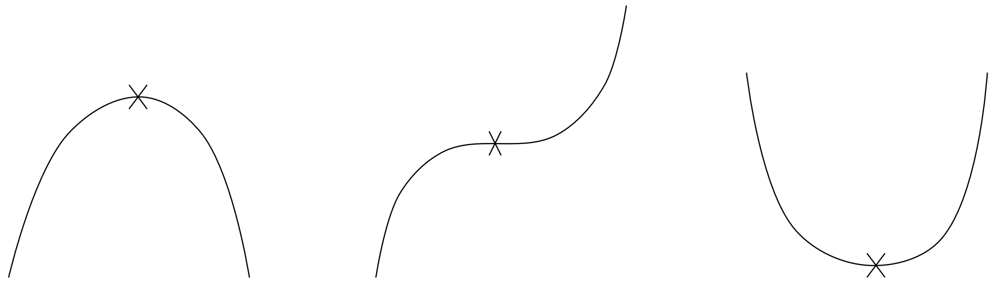
$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h}\\ &= \lim_{h\to 0}\frac{\{x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+h^n\}-x^n}{h}\\ &= \lim_{h\to 0}\left\{nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\cdots+h^{n-1}\right\}\\ &= nx^{n-1}.\end{aligned}$$

i.e.

$$f(x)=x^n, \quad \text{then} \quad f'(x)=nx^{n-1}. \tag{2.4}$$

We could now go on and find derivatives of "all" algebraic functions by definition. But it is too time consuming and impractical.

Important: If $f'(\alpha)=0$, then the tangent to the curve f at $x=\alpha$ is parallel to the x-axis.



(a) *Local maximum.* (b) *Point of inflection.* (c) *Local minimum.*

Figure 2.6: *Different options for when $f'(x)=0$.*

Often, f will have a local minimum or maximum at some $x=\alpha$.

2.1.2 Rules for differentiation

Some simple functions: x^a, a^x, $\sin x$, $\cos x$.

Complicated functions can be derived from these simple ones, by addition, multiplication and composition.

Example 2.6.

$$x + x^2, \quad xa^x, \quad x \sin x, \quad x^3 - x^4 = x^3 + (-x^4), \quad \frac{\sin x}{x} = \frac{1}{x} \cdot \sin x, \quad \cos(x^2).$$

So it is too time consuming to differentiate each individual function we can think of by first principle, i.e. using the definition.

Instead, we want to build a machine to help us differentiate various functions. The machine should contain three bits:

1. Sum rule,
2. Product rule,
3. Chain rule.

The idea of the machine is to tell us how to differentiate functions which are built from simpler pieces as long as we know how to differentiate the smaller pieces.

BONUS: you can still relate the derivative of the entire function to those of the smaller pieces, even if you don't know what the small pieces are.

The sum rule:

If f and g are differentiable, then

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right) = f'(x) + g'(x)$$

Example 2.7. Consider the function $f(x) = \left(x^3 + x^4\right)$, then using the above we have

$$\frac{d}{dx}\left(x^3 + x^4\right) = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^4\right) = 3x^2 + 4x^3.$$

If you repeatedly apply the sum rule, you have

$$\frac{d}{dx}\left(f_1(x) + f_2(x) + \cdots + f_n(x)\right) = \frac{d}{dx}\left(f_1(x)\right) + \frac{d}{dx}\left(f_2(x)\right) + \cdots + \frac{d}{dx}\left(f_n(x)\right).$$

The product rule:

If f and g are differentiable, then

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x). \tag{2.5}$$

Example 2.8.

$$\begin{aligned}\frac{d}{dx}\left[(x^2+1)(x^3-1)\right] &= 2x(x^3-1) + (x^2+1)3x^2 \\ &= 2x^4 - 2x + 3x^4 + 3x^2 \\ &= 5x^4 + 3x^2 - 2x.\end{aligned}$$

Here we have put

$$f(x) = x^2 + 1 \qquad \Longrightarrow \qquad f'(x) = 2x,$$

and

$$g(x) = x^3 - 1 \qquad \Longrightarrow \qquad g'(x) = 3x^2.$$

Example 2.9. Consider the derivative of x^5, so

$$\begin{aligned} \frac{d}{dx}\left(x^5\right) &= \frac{d}{dx}\left(x^4 \cdot x\right) \\ &= 4x^3 \cdot x + x^4 \cdot 1 \\ &= 5x^4, \end{aligned}$$

as expected, since

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

Here we have put

$$f(x) = x^4 \qquad \Longrightarrow \qquad f'(x) = 4x^3,$$

and

$$g(x) = x \qquad \Longrightarrow \qquad g'(x) = 1.$$

This shows that differentiation can be approached in different ways, using what you feel most confident with.[2]

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[2] End Lecture 6.

A special case: if $g(x) = c$, constant, then $g'(x) = 0$, so

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x)) = cf'(x).$$

Therefore the sum rule can be generalised as: If $f_1, f_2, \ldots, f_n$ are differentiable and $a_1, a_2, \ldots, a_n$ are constants, then

$$\frac{d}{dx}\left[a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)\right] = a_1 f_1'(x) + a_2 f_2'(x) + \cdots + a_n f_n'(x)$$

Example 2.10. Consider a polynomial of degree n with constant coefficients, i.e.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n,$$

then

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 \cdots + n a_n x^{n-1},$$

a polynomial of degree $n - 1$, with constant coefficients.

The chain rule:

The chain rule tells us how to differentiate "compositions" of functions. If we have two functions f and g, the composition, denoted by $f \circ g$ (name of new function), is the function given by

$$f \circ g(x) = f(g(x)), \qquad \text{(Do } g \text{ then } f\text{).} \tag{2.6}$$

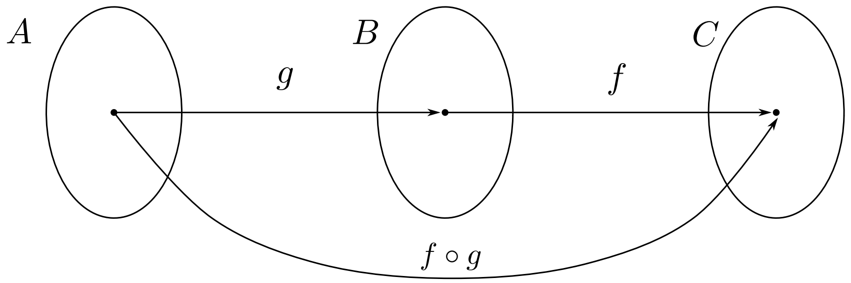


Figure 2.7: *The composition $f \circ g$ first employs g from A to B, then f from B to C.*

Example 2.11. If $f(w) = w^2 + 1$ and $g(u) = \sqrt{u}$, then

$$f \circ g(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1,$$

$$g \circ f(x) = g(f(x)) = g(x^2 + 1) = \sqrt{x^2 + 1}.$$

Here, $f : \mathbb{R} \to \mathbb{R}+$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $f \circ g : \mathbb{R}^+ \to \mathbb{R}^+$, $g \circ f : \mathbb{R} \to \mathbb{R}^+$.

But actually the function $h(x) = x + 1$ can be defined as $h : \mathbb{R} \to \mathbb{R}$, so be careful when generating a function by composition, take note of the difference between $h(x)$ and $f \circ g$ in this case.

In general, $f \circ g \neq g \circ f$.

Composition can be generalised further for more functions, for example suppose we have three functions f, g and h, then

$$f \circ g \circ h(x) = f(g(h(x))).$$

If f and g are differentiable, then

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x). \tag{2.7}$$

Example 2.12. Consider the function $y(x) = (x^3+2x)^{10}$. Here we will choose $f(w) = w^{10}$ and $g(x) = x^3 + 2x$, (so $f'(w) = 10w^9$ and $g'(x) = 3x^2 + 2$). Then

$$\begin{aligned}
\frac{d}{dx}\left(y(x)\right) &= \frac{d}{dx}\left(f(g(x))\right) \\
&= \frac{d}{dx}\left((x^3+2x)^{10}\right) \\
&= f'(g(x))g'(x) \\
&= 10(x^3+2x)^9 \cdot (3x^2+2).
\end{aligned}$$

Essentially, what we have done is to substitute $g(x) = x^3 + 2x$ in our function for $y(x)$, to make the differentiation easier.

Example 2.13. Consider the function $y(x) = 1/x^3$. We know how to differentiate $1/x$. So let us choose $g(x) = x^3$ and $f(w) = 1/w$. Therefore we have $f \circ g(x) = y(x)$. We know the derivatives of f and g are $f'(w) = -1/w^2$ and $g'(x) = 3x^2$. So

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x^3}\right) &= f'(g(x))g'(x) \\
&= -\frac{1}{(x^3)^2} \cdot 3x^2 \\
&= -\frac{3}{x^4},
\end{aligned}$$

i.e. we have

$$\frac{d}{dx}(x^{-3}) = -3x^{-4}.$$

RECALL: we have seen that if n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

e.g.

$$\frac{d}{dx}(x^{365}) = 365x^{364}.$$

If n is a negative integer, then $m = -n$ is a positive integer. So

$$\begin{aligned}\frac{d}{dx}(x^n) &= \frac{d}{dx}\left(\frac{1}{x^m}\right)\\ &= \frac{d}{dx}(f(g(x)))\\ &= f'(g(x))g'(x)\\ &= -\frac{1}{(x^m)^2}\cdot mx^{m-1}\\ &= -mx^{m-1-2n}\\ &= (-m)x^{(-m)-1}\\ &= nx^{n-1}.\end{aligned}$$

Here we have simply chosen $g(x) = x^m$ and $f(w) = 1/w$, where $g'(x) = mx^{m-1}$ and $f'(w) = -1/w^2$.

Therefore we now know that

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

is true for any whole number $n \in \mathbb{Z}$.

What about when $n = 1/2$ i.e. $f(x) = x^{\frac{1}{2}}$. How can we differentiate this? First let us think about what we know about $x^{\frac{1}{2}}$.

We know that $(x^{\frac{1}{2}})^2 = x$!

Let us consider the function $g(y) = y^2$ and take the composition of f and g, that is

$$g \circ f(x) = g(f(x)) = (x^{\frac{1}{2}})^2 = x.$$

In this case f and g are inverse of one another. What does it mean for g to be the inverse of f or f to be the inverse of g?

Aside (NFE):

If we take a point $x = a$ in the domain of f say, and it takes the value $b = f(a)$ in the range (or *image*). Then the inverse function takes the image point b and sends it back to the point $x = a$. In other words we return ourselves back to where we started. There is a well defined rule that goes from a to b and a well defined rule that takes b to a.

The inverse is usually written as f^{-1}, this is just notation.

$$f^{-1}(f(x)) = x \quad \text{then} \quad \begin{cases} f : A \to B \\ f^{-1} : B \to A \end{cases}$$

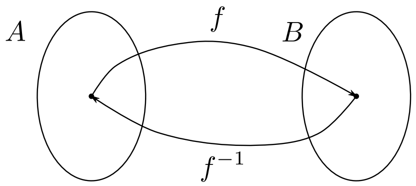


Figure 2.8: *The composition $f \circ f^{-1} = f^{-1} \circ f$ takes you back to where you started!*

Definition 2.2. *The function f^{-1} is called the inverse function for a well defined function f then*

$$f \circ f^{-1}(x) = f^{-1} \circ f(x) = x. \tag{2.8}$$

Not all functions possess inverses. For example the function $f(x) = c$, constant.

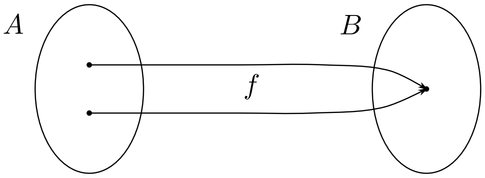


Figure 2.9: *Multi-valued functions do not have inverses (obvious from picture).*

A function is called a *one-to-one* function if it never takes the same value twice, that is

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

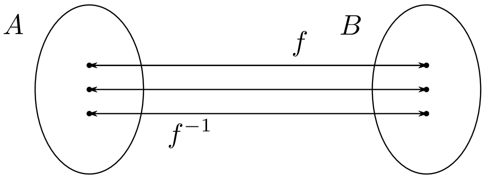


Figure 2.10: *One-to-one functions have inverses (obvious from picture).*

Only for a one-to-one function f, then f^{-1} exists.

Example 2.14. $f(x) = x^2$, if $f : (-\infty, \infty) \to [0, \infty)$ then f^{-1} does not exist since for each $f(x)$, there are two possible x values corresponding to it.

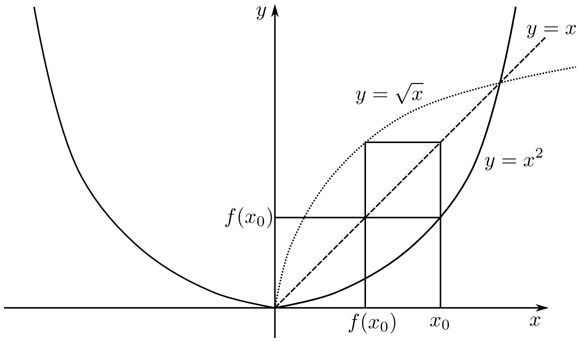


Figure 2.11: *Graph showing the relationship between $y = x^2$ and it's inverse.*

(End NFE)

If it was that $f : [0, \infty) \to [0, \infty)$ i.e. considering the positive x-axis only, then f^{-1} exists. Which in this case we call $f^{-1}(x) = g(x) = \sqrt{x}$, since

$$f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x.$$

And we also know that

$$g(f(x)) = g(x^2) = \sqrt{x^2} = x,$$

i.e. f and g are inverse of one another.

If you draw a function and its inverse on the same coordinate plane, they must be symmetrical about the line $y = x$. Why? Rotate the xy-plane 90° anticlockwise and then flip across the vertical axis. This is because we want the inverse function f^{-1} who's range is the domain of f and vice-versa. Also, equivalent to switch $x \leftrightarrow y$ in $y = f(x)$, rearranging the equation for y to give the inverse.

Now let us return to take the derivative of the function $g(f(x)) = (x^{\frac{1}{2}})^2 = x$, thus

$$\begin{aligned} \frac{d}{dx}\left[g(f(x))\right] &= \frac{d}{dx}(x) \\ \therefore \quad g'(f(x))f'(x) &= 1. \end{aligned}$$

Now since $g'(y) = 2y$, we have

$$2 \cdot f(x)f'(x) = 1.$$

Finally rearranging for $f'(x)$ we see that

$$f'(x) = \frac{1}{2f(x)} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2}x^{-\frac{1}{2}},$$

i.e. the method is the same for integer n (as shown on pg. 23), in words, “bring the power down, reduce the power by one”.[3]

---

[3] End Lecture 7.

So we have proven that

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

is true when n is any integer and also when $n = 1/2$.

It turns out that for all possible powers of p:

$$\frac{d}{dx}(x^p) = px^{p-1}.$$

WARNING: If p is a fraction, such as $p = 1/2$, then we require $x > 0$. We don't want to take the root of negative numbers, i.e. $\sqrt{\text{-m}}$, where $m > 0$.

Example 2.15. Consider the functions $g(x) = x^4 - x^2$, $f(u) = u^{-\frac{4}{3}}$ and so $g'(x) = 4x^3 - 2x$, $f'(u) = -\frac{4}{3}u^{-\frac{7}{3}}$. The composition gives $f \circ g(x) = (x^4 - x^2)^{-\frac{4}{3}}$. So

$$\begin{aligned} \frac{d}{dx}\left[(x^4 - x^2)^{-\frac{4}{3}}\right] &= f'(g(x))g'(x) \\ &= -\frac{4}{3}(x^4 - x^2)^{-\frac{7}{3}}(4x^3 - 2x). \end{aligned}$$

Generalisation: Suppose we have three functions f, g and h. Then the composition is

$$f \circ g \circ h(x) = f(g(h(x))),$$

and it's derivative is

$$\frac{d}{dx}\left[f(g(h(x))\right] = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

Example 2.16. Consider the function $y(x) = \left[(x^3 + x)^{\frac{1}{2}} + 1\right]^{\frac{1}{3}}$. The derivative is

$$\begin{aligned} \frac{d}{dx}\left[(x^3 + x)^{\frac{1}{2}} + 1\right]^{\frac{1}{3}} &= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) \\ &= \frac{1}{3}\left[(x^3 + x)^{\frac{1}{2}} + 1\right]^{-\frac{2}{3}} \cdot \frac{1}{2}(x^3 + x)^{-\frac{1}{2}} \cdot (3x^2 + 1). \end{aligned}$$

Here we chose the functions in the composition $y(x) = f \circ g \circ h(x)$ as follow:

$$\begin{aligned} h(x) = x^3 + x, \quad &\Longrightarrow \quad h'(x) = 3x^2 + 1, \\ g(u) = u^{\frac{1}{2}} + 1 \quad &\Longrightarrow \quad g'(u) = \frac{1}{2}u^{-\frac{1}{2}}, \\ f(w) = w^{\frac{1}{3}} \quad &\Longrightarrow \quad f'(w) = \frac{1}{3}w^{-\frac{2}{3}}. \end{aligned}$$

The composition works as

$$f(g(h(x))) = f(g(x^3 + x)) = f((x^3 + x)^{\frac{1}{2}} + 1) = \left[(x^3 + x)^{\frac{1}{2}} + 1\right]^{\frac{1}{3}}.$$

Example 2.17. Now consider the example where $y(x) = \left[(x^3 + x)^{\frac{1}{2}} + x\right]^{\frac{1}{3}}$. It is similar to the above example, however, when replacing "1" by "x" in the square bracket, finding a nice composition becomes a little harder. Instead of trying to work out what this function is as a composition, we simply apply the chain rule using the method "differentiate outer bracket, work inwards". When the derivative is performed on $y(x)$ it follows the steps:

$$\begin{aligned}
\frac{d}{dx}\left[(x^3+x)^{\frac{1}{2}}+x\right]^{\frac{1}{3}} &= \frac{1}{3}\left[(x^3+x)^{\frac{1}{2}}+x\right]^{\frac{1}{3}-1}\frac{d}{dx}\left[(x^3+x)^{\frac{1}{2}}+x\right]\\
&= \frac{1}{3}\left[(x^3+x)^{\frac{1}{2}}+x\right]^{-\frac{2}{3}}\left[\frac{1}{2}(x^3+x)^{-\frac{1}{2}}\frac{d}{dx}(x^3+x)+1\right]\\
&= \frac{1}{3}\left[(x^3+x)^{\frac{1}{2}}+x\right]^{-\frac{2}{3}}\left[\frac{1}{2}(x^3+x)^{-\frac{1}{2}}(3x^2+x)+1\right].
\end{aligned}$$

Note that in addition to the chain rule, the sum rule has also been applied.

Finally, there is another way of writing the chain rule (or other derivatives). Let $w = g(x)$, $y = f(w) = f(g(x))$. Then we can write

$$\frac{dy}{dx} = \frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x) = \frac{dy}{dw}\cdot\frac{dw}{dx}.$$

Machine:

we have gained 3 basic rules of differentiation,

1. sum rule:
$$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{df}{dx}+\frac{dg}{dx},$$
2. product rule:
$$\frac{d}{dx}\left(f(x)g(x)\right) = g(x)\frac{df}{dx}+f(x)\frac{dg}{dx},$$
3. chain rule:
$$\frac{d}{dx}\left(f(g(x))\right) = \frac{df}{dg}\frac{dg}{dx}.$$

Extras:
$$\frac{d}{dx}(x^p) = px^{p-1}.$$

2.2 Differentiation of trigonometric functions

How do we find the derivative of $f(x) = \sin x$? By definition 2.1 (on pg. 22), the derivative of $f(x) = \sin x$ at the point $x = c$ is

$$\begin{aligned}
\frac{df}{dx}(c) &= \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\\
&= \lim_{h\to 0}\frac{\sin(c+h)-\sin(c)}{h}\\
&= \lim_{h\to 0}\frac{\sin(c)\cos(h)+\sin(h)\cos(c)-\sin(c)}{h}\\
&= \lim_{h\to 0}\left\{\sin(c)\frac{\cos(h)-1}{h}+\cos(c)\frac{\sin(h)}{h}\right\}.
\end{aligned}$$

If we know the limits of $\sin(h)/h$ and $(\cos(h) - 1)/h$ as $h \to 0$, then we will know $f'(c)$, since c is unrelated to h.

First let us consider $\sin(h)/h$. We draw a circular sector with a very small angle, where the curved side is of length x (in radians). For small angle x (in radians), $\sin x$ and x are almost equal, i.e.

$$\frac{\sin x}{x} \approx 1.$$

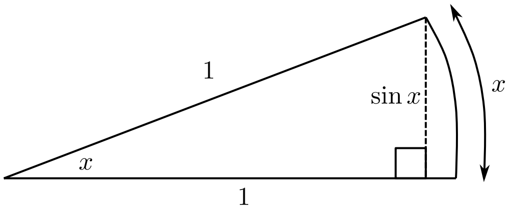
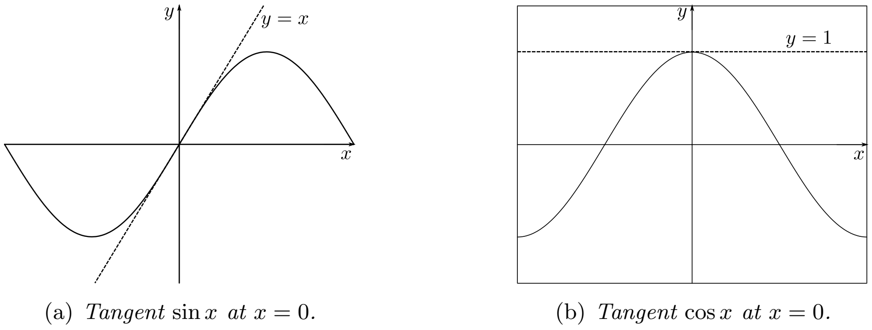


Figure 2.12: *Very small circular sector. Right angled triangle with hypotenuse and adjacent approximately equal i.e. two radii from circular sector.*

As $h \to 0$, $\sin(h)/h \to 1$, if you work in radians. Actually

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(h)}{h} &= \lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h}\\ &= \left.\frac{d}{dx}(\sin x)\right|_{x=0}\\ &= \cos(0)\\ &= 1,\end{aligned}$$

that is, the derivative of $\sin x$ at $x = 0$ is one, or the tangent line of $y = \sin x$ at $x = 0$ is $y = x$.



(a) *Tangent* $\sin x$ *at* $x = 0$.

(b) *Tangent* $\cos x$ *at* $x = 0$.

Figure 2.13: *Examining the tangents at $x = 0$ gives an insight of the derivative at this point.*

Second, we notice that

$$\begin{aligned} \lim_{h\to 0}\frac{\cos(h)-1}{h} &= \lim_{h\to 0}\frac{\cos(0+h)-cos(0)}{h} \\ &= \frac{d}{dx}(\cos x)\bigg|_{x=0} \\ &= \sin(0) \\ &= 0, \end{aligned}$$

i.e. the derivative of $\cos x$ at $x = 0$ is zero, since the tangent line of $y = \cos x$ at $x = 0$ is horizontal ($y = 1$). So

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0.$$

Therefore

$$f'(c) = \cos(c),$$

i.e.,

$$\frac{d}{dx}(\sin x) = \cos x.$$

Similarly, we can derive

$$\frac{d}{dx}(\cos x) = -\sin x.$$

NOTE: The change in sign when differentiating $\cos x$!

Exercise 2.1. Now that we know the derivatives of sin and cos, we should be able to calculate the derivative of $\tan x$. Try it yourself before next lecture![4]

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[4] End Lecture 8.

Since we know the derivatives of sin and cos, we should be able to find the derivative of tan since

$$\tan x = \frac{\sin x}{\cos x}.$$

Before we attempt to differentiate (using the product rule), there is another "so called" rule called the *quotient rule*, really, it is simply the product rule for rational functions. We derive the formula below, however if you do not remember the formula, it's best to work with the product rule.

First we write

$$\frac{f(x)}{g(x)} = f(x)\,[g(x)]^{-1},$$

as a product, then apply the product rule as follows:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = f'(x)\,[g(x)]^{-1} + f(x)\frac{d}{dx}\left\{[g(x)]^{-1}\right\}.$$

We need the derivative of $[g(x)]^{-1}$. This is obtained by the chain rule, if we consider $r(u) = 1/u$, then $r'(u) = -1/u^2$, so

$$\begin{aligned}\frac{d}{dx}[g(x)]^{-1} &= \frac{d}{dx}[r(g(x))] \\ &= r'(g(x))g'(x) \\ &= -\frac{g'(x)}{[g(x)]^2}.\end{aligned}$$

Finally, we have

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = f'(x)\,[g(x)]^{-1} - f(x)\frac{g'(x)}{[g(x)]^2},$$

finding a common denominator, we may write

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \tag{2.9}$$

This is known as the *quotient rule*.

Example 2.18. Consider the function $\tan x$. We will use the quotient rule where $f(x) = \sin x$ and $g(x) = \cos x$, therefore $f'(x) = \cos x$ and $g'(x) = -\sin x$. Hence

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \\ &= 1 + \tan^2 x.\end{aligned}$$

Exercise 2.2. Try find the derivatives of cosec$x$, $\sec x$ and $\cot x$.

Example 2.19. Consider the function $\cos(x^2+2x)$. Let us choose $f(g)=\cos(g)$ and $g(x)=x^2-2x$, i.e. $f'(g)=-\sin(g)$ and $g'(x)=2x-2$. Then, using the chain rule

$$\begin{aligned} \frac{d}{dx}\left[\cos(x^2+2x)\right] &= \frac{d}{dx}\left[f(g(x))\right] \\ &= f'(g(x))g'(x) \\ &= -(2x-2)\cdot(\sin(x^2-2x)). \end{aligned}$$

Example 2.20. Consider the function $\cos^4 x \sin x$. Here we will first apply the product rule, so that

$$\frac{d}{dx}\left[\cos^4 x\sin x\right] = \sin x\frac{d}{dx}(\cos^4 x) + \cos^4 x\frac{d}{dx}(\sin x).$$

The derivative of $\sin x$ is easy, we know this. All we need to do now is apply the chain rule on $\cos^4 x$. Here we will choose $f(u)=u^4$, therefore $f'(u)=4u^3$ and $g(x)=\cos x$, whose derivative is $g'(x)=-\sin x$. Thus we have

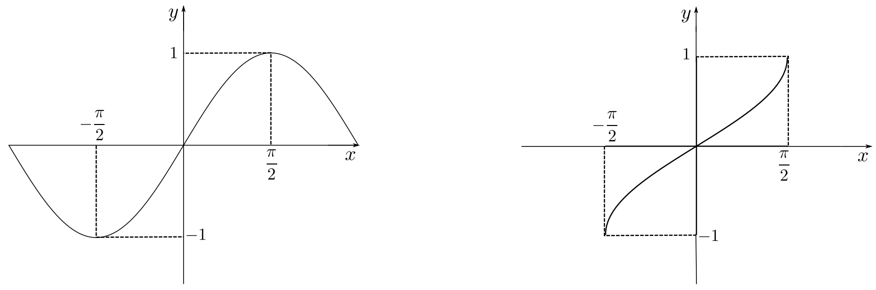
$$\begin{aligned} \frac{d}{dx}\left[\cos^4 x\sin x\right] &= \sin x\frac{d}{dx}(f(g(x))) + \cos^4 x\cdot\cos x \\ &= \sin x(f'(g(x))g'(x) + \cos^5 x \\ &= \sin x\cdot 4\cos^3 x\cdot(-\sin x) + \cos^5 x \\ &= -4\sin^2 x\cos^3 x + \cos^5 x. \end{aligned}$$

2.2.1 The derivatives of inverse trig functions

Now that we know the derivatives of the trigonometric functions, let us consider the inverses of these functions, i.e. $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.

NOTATION: $\sin^{-1} x$ is the inverse function of $\sin x$, whereas $(\sin x)^{-1} = 1/\sin x$.

First let us see how to define $\sin^{-1} x$. To define $\sin^{-1}$, we first must put a constraint on the domain of sin. Recall, a function which associates more than value of x for $f(x)$ cannot have an inverse. So we choose the domain $[-\frac{\pi}{2},\frac{\pi}{2}]$.



(a) *Limiting* $\sin x$.

(b) *Graph of* $\sin^1 x$.

Figure 2.14: *Reflecting* $\sin x$ *in* $y=x$ *gives* $\sin^{-1} x$. $\sin : [-\frac{\pi}{2},\frac{\pi}{2}] \to [-1,1]$ *and* $\sin^{-1} : [-1,1] \to [-\frac{\pi}{2},\frac{\pi}{2}]$.

Let $f(u) = \sin u$ and $g(x) = \sin^{-1} x$, we need to work out $g'(x)$. We know that $f(g(x)) = x$, so differentiating on both sides we have

$$f'(g(x))g'(x) = 1 \qquad \Longrightarrow \qquad g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\sin^{-1} x)}.$$

If we let $\theta = \sin^{-1} x$, then we have $\sin\theta = x$. Now we construct a right angle triangle which satisfies $\sin\theta = x$.

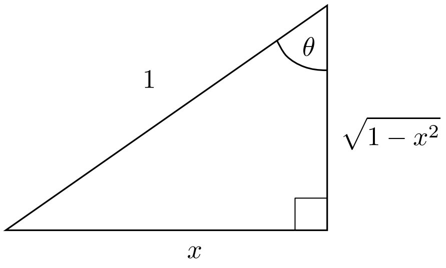


Figure 2.15: *If* $\sin(\theta) = x$, *one can construct such a right angle triangle using Pythagorus' theorem.*

Then we know from the triangle that

$$\cos\theta = \sqrt{1-x^2}.$$

Therefore

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

In the same way, we can work out

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}.$$

Exercise 2.3. Define $g(x) = \cos^{-1} x$ first and then work out $g'(x)$.[5]

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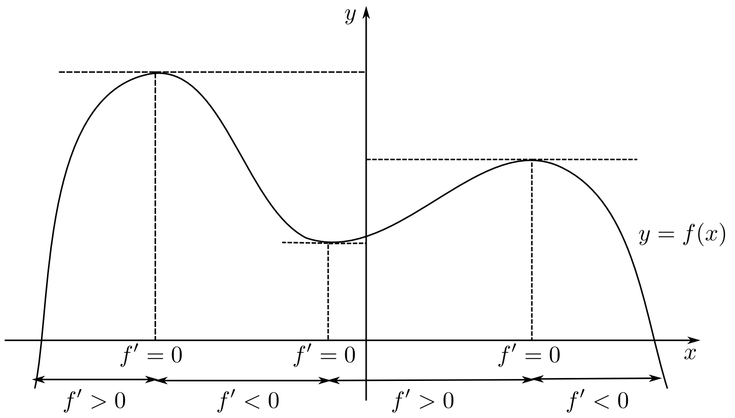
[5] End Lecture 9.

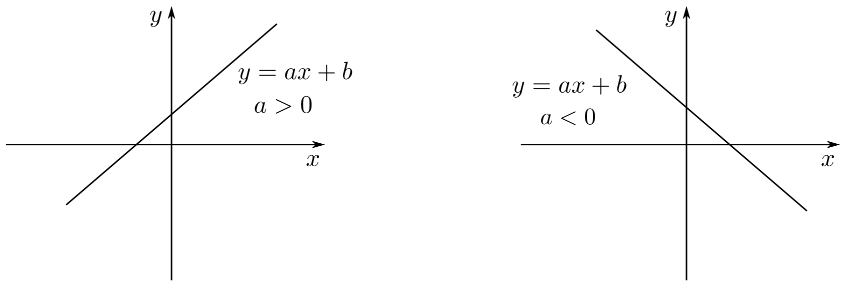
2.3 Maxima, minima and second derivatives

Consider the following question: given some function f, where does it achieve its maximum or minimum values?

First let us examine in more detail what $f'(x)$ tells us about $f(x)$.



(a) *Maxima and minima of some graph,* $y = f(x)$*.*



(b) *Positive slope.*

(c) *Negative slope.*

Figure 2.16: *Displaying the change in first derivative before, at and after maxima and minima.*

We notice that:

If $f(x)$ is increasing, then $f'(x) > 0$,

If $f(x)$ is decreasing, then $f'(x) < 0$.

Therefore, troughs and humps occur at places through which f' changes sign, i.e. when $f' = 0$, where

$$\text{Trough} = \text{local minimum}, \quad \text{Hump} = \text{local maximum}.$$

The derivative gives us a way of finding troughs and humps, and so provides good places to look for maximum and minimum values of a function.

Example 2.21. Find the maximum and minimum values of the function

$$f(x) = x^3 - 3x, \quad \text{on the domain } -\frac{3}{2} \leq x \leq \frac{3}{2}.$$

Differentiating $f(x)$ we have

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1).$$

Hence, at $x = \pm 1$, we have $f'(x) = 0$. The value of the function at these points are

$$f(1) = -2, \quad f(-1) = 2.$$

It also maybe possible for the function f to achieve its maximum or minimum at the ends of the domain. Therefore we calculate

$$f\left(\frac{3}{2}\right) = -\frac{9}{8}, \quad f\left(-\frac{3}{2}\right) = \frac{9}{8}.$$

So comparing the four values of f, we know where $f(x)$ is biggest or smallest. So we finally conclude

$$f(1) = -2 \text{ (min)}, \quad f(-1) = 2 \text{ (max)},$$

i.e. we have a minimum at the point $(1, -2)$ and a maximum at $(-1, 2)$.

If we had been looking in the range $-3 \leq x \leq 3$, then at the ends

$$f(3) = 18 \text{ (max)}, \quad f(-3) = -18 \text{ (min)}.$$

To draw the graph of f, we need to find the values of f at some important points, such as when $x = 0$, when $f(x) = 0$ and $f'(x) = 0$. We already know the points at which $f'(x) = 0$.

When $x = 0$ we have $f(0) = 0$, i.e. the graph passes through the point $(0, 0)$.

When $f(x) = 0$, we need to solve the equation

$$x^3 - 3x = x(x^2 - 3x) = 0.$$

So either $x = 0$ (we already have this point), or

$$x^2 - 3 = 0 \quad \Longrightarrow \quad x = \pm\sqrt{3}.$$

So the graph also passes through the points $(0, \sqrt{3})$ and $(0, -\sqrt{3})$.

Now let us examine the derivative further, $f'(x) = 3x^2 - 3$.

$$\begin{aligned} f'(x) > 0 \quad &\text{when} \quad x < -1, \\ f'(x) \leq 0 \quad &\text{when} \quad -1 \leq x \leq 1, \\ f'(x) > 0 \quad &\text{when} \quad x > 1. \end{aligned}$$

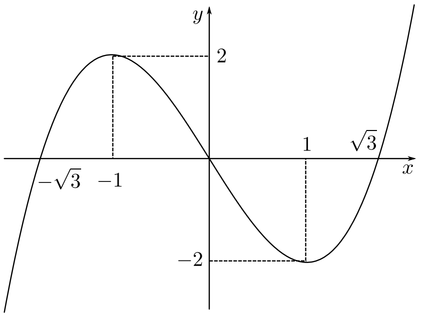


Figure 2.17: *Sketch of* $f(x) = x^3 - 3x$*, displaying turning points (maximum and minimum).*

Sketching graphs, things to remember:

1. Find $f(x)$ when $x = 0$, i.e. where the graph cuts the y-axis.
2. Find x when $f(x) = 0$, i.e. when the graphs cuts the x-axis.
3. Find x when $f'(x) = 0$, i.e. the *stationary* points of the graph. (Plug into f(x) to find y value).
4. Determine the sign of $f'(x)$ on either side of the stationary points to determine weather stationary points are minimum, maximum or points of inflection.

Basic principles:

Let $f : [a, b] \to \mathbb{R}$. If f achieves a local maximum or local minimum at x, then either:

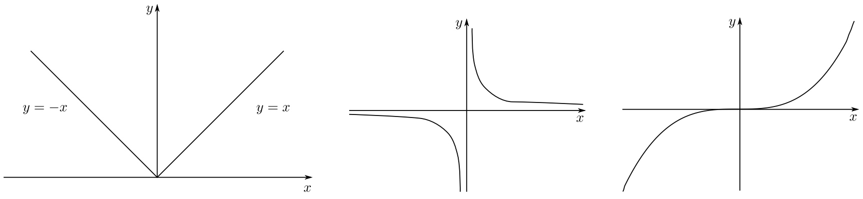
(i) $f'(x) = 0$, or
(ii) $x = a$, or
(iii) $x = b$, or
(iv) where $f'(x)$ doesn't exist.

So to find the local maximum/minimum of f, it suffices to list possibilities in (i)-(iv) and then check.

Example 2.22. Consider $f(x) = |x|$, on the domain $-1 \leq x \leq 1$. We know $f'(0)$ doesn't exist, but at $x = 0$, $f(x)$ achieves its minimum value of 0.

Example 2.23. Consider $f(x) = 1/x$ on the domain $-2 \leq x \leq 2$. There is no maximum or minimum on this range. $f'(x) = -1/x^2$ is not defined at $x = 0$ (it actually tends to $\pm\infty$ either side of $x = 0$).

Example 2.24. Consider $f(x) = x^3$ on the domain $-2 \leq x \leq 2$. $f'(x) = 3x^2$, which is zero if $x = 0$. So zero is a stationary point for x^3, but it is neither a hump or a trough. It's a point of inflection.



(a) $y = |x|$ (Ex. 2.22). (b) $y = \frac{1}{x}$ (Ex. 2.23). (c) $y = x^3$ (Ex. 2.24).

Figure 2.18: *Different options for when* $f'(x) = 0$.

2.3.1 Second derivative

To characterise troughs and humps (local maximums and minimums), we need the knowledge of second derivatives.

If we start with some function, say $f(x) = x^2$, we know the derivative $f'(x)$ is well defined for every x, and $f'(x) = 2x$. We can view $f'(x)$ itself as a function, so we may differentiate it again to have

$$\frac{d}{dx}(f'(x)) = \frac{d^2}{dx^2}(f(x)) = f''(x) = 2.$$

In this way, we can define f'', $f^{(3)}$, $f^{(4)}$ and so on.

Example 2.25.

$f(x) = x^n$, n is a positive whole number.

$f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^{n-2}$,

$f^{(3)}(x) = n(n-2)(n-2)x^{n-3}, \ldots$

$f^{(n)}(x) = n(n-1)(n-2)\ldots 2 \cdot 1 = n!$, recall $! \equiv$ Factorial.

$f^{(m)}(x) = 0$, $m > n$.

In physics, if $f(t)$ represents distance as a function of time, then $f'(t)$ represents speed (i.e. the rate of change in distance), and $f''(t)$ represents acceleration, i.e. the rate of change of speed. This is the key to understanding how things move, in particular under gravity.

The geometric interpretation of f'':

1. If $f'' > 0$, then the slope of the tangent line is increasing in value (from left to right), so possible shapes for $f(x)$ are like:

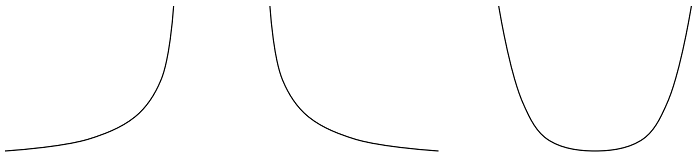


Figure 2.19: *Slope of tangent increases in value (possibly negative to positive).*

Therefore if $f'(c) = 0$ and $f''(c) > 0$, then around c, $f(x)$ is a trough and we can expect a local minimum value of f at $x = c$.

Example 2.26.

$$\begin{aligned} f(x) &= x^3 - 3x, \\ f'(x) &= 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1), \\ f''(x) &= 6x. \end{aligned}$$

At $x = 1$, $f'(x) = 0$, $f''(x) = 6 > 0$, so we have a trough at $x = 1$.

2. If $f'' < 0$, then the slope of the tangent line is decreasing (from left to right), so possible shapes for $f(x)$ are like:

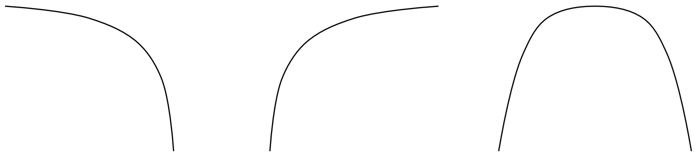


Figure 2.20: *Slope of tangent decreases in value (possibly positive to negative) .*

Therefore if $f'(c) = 0$ and $f''(c) < 0$, then around c, $f(x)$ is a hump, and we expect a local maximum value of f at c.

Example 2.27. Continuing with $f(x) = x^3 - 3x$, at $x = -1$ we have $f'(-1) = 0$, $f''(-1) = -6 < 0$.

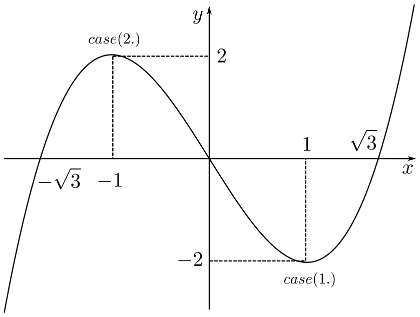


Figure 2.21: *Characterising the turning points using second derivatives of* $y = x^3 - 3x$.

3. If $f'(c) = 0$ and $f''(c) = 0$, then the slope often doesn't change sign, i.e. it goes from positive slope to zero to positive slope (decreasing to zero then increasing), or negative to zero to negative (increasing to zero then decreasing). These points are points of inflection and possible shapes are like:

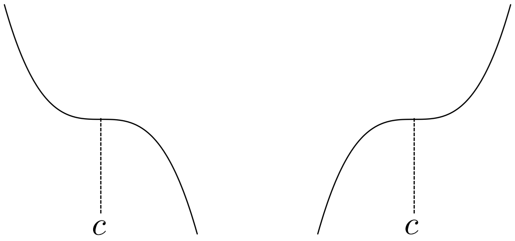


Figure 2.22: *Points of inflection at c, slope goes from positive to positive or negative to negative.*

Example 2.28. $f(x) = x^3$, so $f'(x) = 3x^2$ and $f''(x) = 6x$. At $x = 0$ we have $f'(0) = f''(0) = 0$.

WARNING: However, $f(x)$ doesn't necessarily need to be a point of inflection when the second derivative is zero.

Example 2.29. Consider $f(x) = x^4$. Now we have $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Notice that $f'(0) = f''(0) = 0$, however when we sketch $f(x)$, we realise it has a local minimum at $x = 0$.

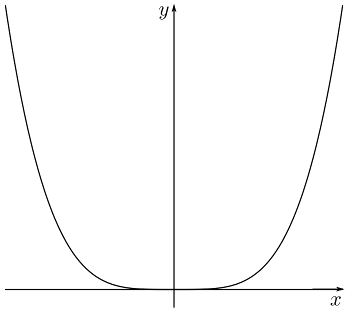


Figure 2.23: *Graph of $y = x^4$, clearly doesn't have an inflection point at $x = 0$, yet $f''(0) = 0$.*

Therefore we say that if f has a point of inflection at $x = c$, then $f'(c) = f''(c) = 0$ is a necessary condition.

Example 2.30. (A slightly more advanced example). Consider the function

$$f(x) = \frac{x^2 - 4}{x^2 - 1} = \frac{(x-2)(x+2)}{(x-1)(x+1)}.$$

So $f(x) = 0$ when the numerator of $f(x)$ is equal to zero. That is

$$(x-2)(x+2) = 0 \quad \implies x = 2,\, x = -2.$$

Something strange happens at the points $x = 1$ and $x = -1$. We get zero in the denominator and the numerator is negative in both cases so $f(\pm 1) \to -\infty$.

The graph cuts the y-axis when $x = 0$, that is

$$f(0) = \frac{-4}{-1} = 4.$$

Now let us calculate the derivative, it is a rational function so we may use the quotient rule (or apply the product rule), so we have

$$f'(x) = \frac{2x(x^2-1) - 2x(x^2-4)}{(x^2-1)^2} = \frac{6x}{(x^2-1)^2}.$$

So $f'(x) = 0$ when the numerator is zero i.e. when $6x = 0$, thus there is only one turning point at $x = 0$.

Now let us differentiate $f'(x)$, again using the quotient rule we have

$$f''(x) = \frac{6(x^2-1)^2 - 6x(2 \cdot (x^2-1) \cdot 2x)}{(x^2-1)^4} = \frac{6(x^2-1)^2 - 24x^2 \cdot (x^2-1)}{(x^2-1)^4}.$$

It remains to check the value of f'' at the turning point $x = 0$. Therefore

$$f''(0) = \frac{6(-1)^2 - 0}{(-1)^4} = 6 > 0,$$

and we have a local minimum at this point.

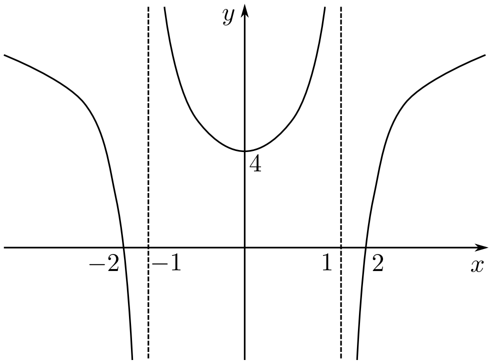


Figure 2.24: *Graph of* $y = (x^2 - 4)/(x^2 - 1)$, *i.e. a rational function with only one turning point (minimum) at* $x = 0$.

At $x = \pm 1$, we have what is known as *asymptotes*, lines the graph never touches, but gets very close to as we approach plus or minus infinity.[6]

---

[6] End Lecture 10.

2.4 Real life examples

Example 2.31. You are required to design an open box with a square base and a total volume of 4m^3 using the least amount of materials.

We are hunting for particular dimensions of the box. Lets give them names, so we will call the length of the base edges a and let the hight of the box be called b. See Fig. 2.25.

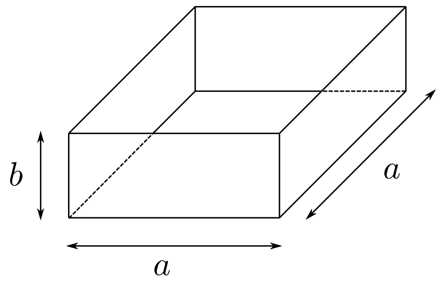


Figure 2.25: *Square based open top box.*

In terms of a and b, the volume of the box is

$$a^2 b = 4 \ (\text{m}^3).$$

We want to minimise the amount of material, i.e. the question is how much do we use? Consider what we know about the box. We have a base of area a^2 and 4 sides, each with area ab. So the total area is

$$a^2 + 4ab.$$

We want to find the minimum value of $a^2 + 4ab$, subject to the condition $a^2 b = 4$ and more obviously $a > 0$, $b > 0$. From the condition, we see that $b = 4/a^2$, so we can rewrite the amount of material in terms of a:

$$a^2 + 4ab = a^2 + 4a\left(\frac{4}{a^2}\right) = a^2 + \frac{16}{a}.$$

So now we can write a function for the material area solely in terms of one of the lengths of the box (the other is now fixed by using the condition). That is

$$f(a) = a^2 + \frac{16}{a}.$$

Now written like a function, it is easy to see how we would minimise the material, that is by minimising the function $f(a)$. So we have to calculate $f'(a)$, which is

$$f'(a) = 2a - \frac{16}{a^2},$$

and now we simply need to see when $f'(a) = 0$. Finally, we have

$$2a - \frac{16}{a^2} = 0 \qquad \Longrightarrow \qquad 2a^3 = 16 \qquad \Longrightarrow \qquad a = 2.$$

The only real number at which $f'(a) = 0$ is $a = 2$. The function $f'(a)$ makes sense except at $a = 0$, which is outside the range (since $a > 0$). In addition, in this situation we don't have a closed domain (i.e. no ends) since $0 < a < +\infty$. We can differentiate $f'(a)$, which gives

$$f''(a) = 2 + \frac{32}{a^3}.$$

This is positive for the whole range (and at $a = 2$), along with sketching the graph in Fig. 2.26, we can mathematically confirm choosing such an a would use minimal materials.

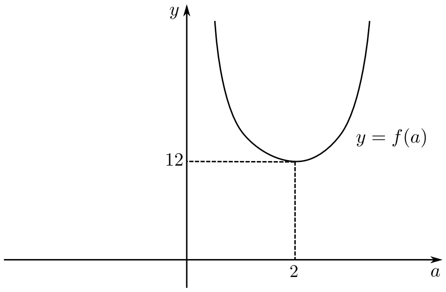


Figure 2.26: *Graph of* $f(a) = a^2 + 16/a$*, displaying changes in area of material required for different choices of length* a*, given fixed volume condition on* b*.*

NOTATION: For an open domain, for example $a < x < b$, we use curly brackets, i.e. $a \in (a, b)$. For an closed domain, for example $a \leq x \leq b$, we use square brackets, i.e. $a \in [a, b]$.

Example 2.32. We have a pair of islands, island 1 and island 2, 20Km and 10Km away from a straight shore, respectively. The perpendiculars from the islands to the shore are 30Km apart (along the shore). What is the quickest way between the two islands that goes via the shore?

We are trying to find a point along the shore, which we want to visit when going from one island to the other. This point can specified by the distance from the perpendicular of island 1, call this distance x.

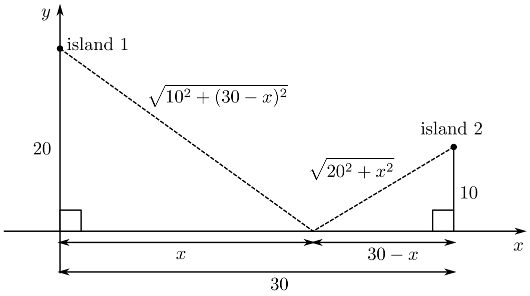


Figure 2.27: *The set up of islands relative to the shore (x-axis).*

The total distance $D(x)$ to be travelled is given by the formula

$$D(x) = \sqrt{20^2+x^2} + \sqrt{10^2+(30-x)^2}.$$

On geometric grounds, we see that $0 \le x \le 30$, and $D(x)$ is differentiable in this range.

$$\begin{aligned} D'(x) &= \frac{1}{2}(20^2+x^2)^{-\frac{1}{2}}2x + \frac{1}{2}\left[10^2+(30-x)^2\right]^{-\frac{1}{2}}2(30-x)(-1) \\ &= \frac{x}{\sqrt{20^2+x^2}} - \frac{30-x}{\sqrt{(30-x)^2+10^2}}. \end{aligned}$$

To find the minimum distance we set $D'(x)=0$, so we have

$$\frac{x}{\sqrt{20^2+x^2}} = \frac{30-x}{\sqrt{(30-x)^2+10^2}}.$$

Squaring both sides we get

$$\frac{x^2}{20^2+x^2} = \frac{(30-x)^2}{(30-x)^2+10^2} \qquad \Longrightarrow \qquad (x^2)[(30-x)^2+10^2] = (30-x)^2(20^2+x^2)$$

Expanding the brackets we see

$$10^2x^2 = 20^2(30-x)^2.$$

Take the square root of both sides of the last equation gives

$$10x = \pm 20(30-x).$$

Since $30-x \ge 0$, "-" sign is not possible, so

$$10x = 20(30-x) \qquad \Longrightarrow \qquad x = 20.$$

The only possible places where minimum can occur are

$$x=0, \quad x=20, \quad x=30,$$

where

$$D(0) = 20 + \sqrt{10^2+30^2} \approx 51.6,$$

$$D(20) = 20\sqrt{2} + 10\sqrt{2} = 30\sqrt{2} \approx 42.4,$$
$$D(30) = \sqrt{20^2 + 30^2} + 10 \approx 46.06.$$

So the minimum value does occur at $x = 20$.

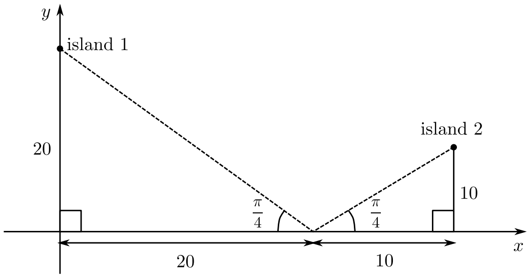


Figure 2.28: *Optimum route from island 1 to island 2, arriving and leaving shore at an angle of $\pi/4$ radians.*

The optimal route is to leave the shore at the same angle of arrival.

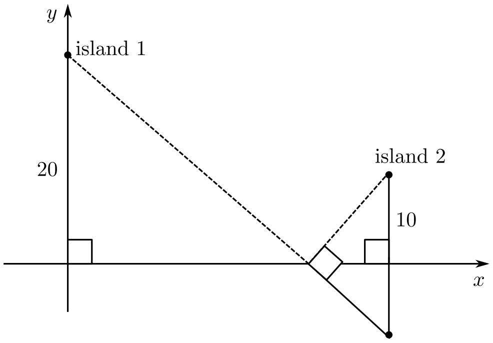


Figure 2.29: *Sketch of alternative method of calculating shortest distance between the islands, using simple geometric properties, gaining same result.*

Could we have deduced that this route was the shortest without calculus? Yes! We could have reflected island 2 in the shore line to obtain an imaginary island. Then it is easy to see that the shortest route from island 1 to the imaginary island is a straight line.[7]

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[7] End Lecture 11.

Chapter 3

Exponentials and Logarithms

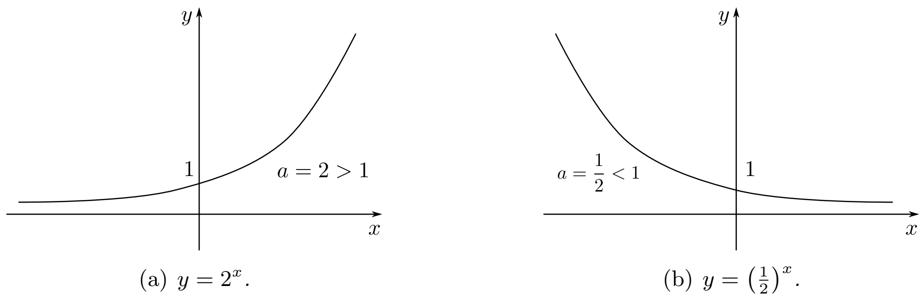
3.1 Exponentials

An exponential function is a function of the form

$$f(x) = a^x,$$

where a is a positive constant.

Example 3.1.



(a) $y = 2^x$.

(b) $y = \left(\frac{1}{2}\right)^x$.

Figure 3.1: *The domain:* $-\infty < x < +\infty$*; the range:* $(0, +\infty)$.

For

$a > 1$, $f(x)$ increases as x increases.

$a < 1$, $f(x)$ decreases as x increases.

$a = 1$, $f(x) = 1$.

$a^0 = 1$ for each a, so the graph always passes through the point $(0, 1)$.

3.1.1 Slope of exponentials

First let us consider the slope at $x = 0$.

Example 3.2. Suppose we have $f(x) = 2^x$, then applying the definition of the derivative we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{2^h - 1}{h}.$$

h	$f'(0) \approx$
0.1	0.7177
0.01	0.6955
0.001	0.6933
0.0001	0.6932

So for $f(x) = 2^x$, $(a = 2)$, we have slope ≈ 0.693 at $x = 0$.

Similarly, for $f(x) = 3^x$, $(a = 3)$, we have slope ≈ 1.698 at $x = 0$.

Therefore, we expect that there is a number between 2 and 3 such that the slope at $x = 0$ is 1. This number is called e, where $e \approx 2.718281828459\ldots$. The number e is irrational.

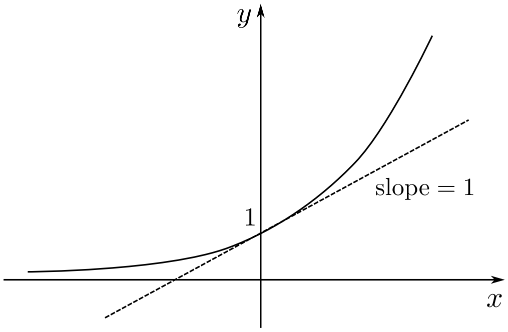


Figure 3.2: *Graph of $y = e^x$, which has tangent with slope of 1 at $x = 1$.*

The fact that the slope is 1 at $x = 0$ tells us that

$$\frac{e^h - e^0}{h} = \frac{e^h - 1}{h} \to 1, \quad ash \to 0.$$

Therefore, if h is small, then $e^h - 1 \approx h$, i.e.

$$e^h \approx 1 + h.$$

We call $f(x) = e^x = \exp(x)$ the *exponential function.*

To find the slope at $x = c$, we need to look at

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0} \frac{e^{c+h} - e^c}{h} = \lim_{h\to 0} \frac{e^c(e^h - 1)}{h} = e^c \lim_{h\to 0} \frac{e^h - 1}{h} = e^c,$$

i.e. the derivative of e^x is itself,

$$\frac{d}{dx}(e^x) = e^x.$$

Example 3.3. Consider $f(x) = e^{\sqrt{1+x}}$. Here we will employ the chain rule. Choose $g(x) = \sqrt{1+x}$ and $f(u) = e^u$, so we have $g'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ and $f'(u) = e^u$.

$$\begin{aligned} \frac{d}{dx}\left(e^{\sqrt{1+x}}\right) &= f'(g(x))g'(x) \\ &= e^{\sqrt{1+x}}\frac{1}{2}(1+x)^{-\frac{1}{2}} \\ &= \frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}}. \end{aligned}$$

3.2 The natural logarithm

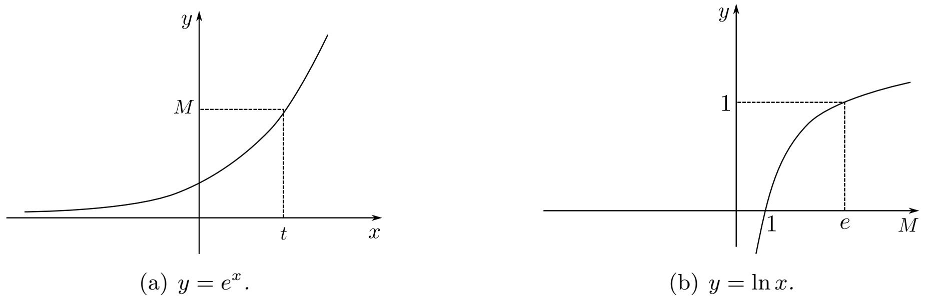
Consider the inverse function of $f(x) = e^x$. In $f(x)$, every positive number occurs as the exponential of something, i.e. $M = e^t$ for an appropriate t. The number t is called $\ln(M)$: the natural logarithm of M. In other words

"$\ln M$ is the number whose exponential is M": $e^{\ln M} = M$.

In this way, we define the function of the natural logarithm

$$g(M) = \ln M.$$

Example 3.4.



(a) $y = e^x$. (b) $y = \ln x$.

Figure 3.3: *For* $\ln x$ *the domain:* $(0, +\infty)$*; the range:* $(-\infty, +\infty)$*. The graph* $y = e^x$ *has a horizontal asymptote at* $y = 0$*, while* $y = \ln x$ *has a vertical asymptote at* $x = 0$*.*

By definition,

$$e^{\ln M} = M, \quad \ln(e^x) = x,$$

which means that if you perform ln(exp) or take the exp(ln), then we get back to where we started.

3.2.1 Characteristic properties

1. $\ln(MN) = \ln M + \ln N$.
2. $\ln(M^p) = p \ln M$.

Logarithms are used among other things to solve "exponential equations".

Example 3.5. Find x, given $3^x = 7$. Taking the logarithm of both sides we have

$$\ln(3^x) = \ln 7 \qquad \Longrightarrow \qquad x \ln 3 = \ln 7.$$

Rearranging we have

$$x = \frac{\ln 3}{\ln 7} \approx \frac{1.95}{1.10} \approx 1.77.$$

Exercise 3.1. Show that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Hint: put $y = \ln x$.

Example 3.6. Consider the function $f(x) = \ln(\cos x)$. Choose $g(x) = \cos x$ and $f(u) = \ln u$, so we have $g'(x) = -\sin x$ and $f'(u) = 1/u$. Thus

$$\begin{aligned} \frac{d}{dx}(\ln(\cos x)) &= f'(g(x))g'(x) \\ &= \frac{1}{\cos x} \cdot (-\sin x) \\ &= -\tan x. \end{aligned}$$

Similarly we have

$$\begin{aligned} \frac{d}{dx}(\sin(\ln x)) &= \cos(\ln x) \cdot \frac{1}{x} \\ &= \frac{\cos(\ln x)}{x}. \end{aligned}$$

Differentiation of other exponentials

In order to differentiate for example 3^x, we must express it in firms of e^x :

$$3 = e^{\ln 3} \qquad \Longrightarrow \qquad 3^x = (e^{\ln 3})^x = e^{x \ln 3}$$

Therefore we calculate the derivative of 3^x as follows:

$$\begin{aligned} \frac{d}{dx}(3^x) &= \frac{d}{dx}(e^{x \ln 3}) \\ &= f'(g(x))g'(x) \\ &= e^{x \ln 3} \cdot \ln 3 \\ &= 3^x \cdot \ln 3. \end{aligned}$$

Here we chose $g(x) = x\ln 3$ and $f(u) = e^u$ so that $g'(x) = \ln 3$ and $f'(u) = e^u$.

In general, for any positive constant a

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

NOTATION: $\log_e x = \ln x$, is the "proper" way of writing the natural logarithm.

3.2.2 Logarithms base a

We can also define $\log_a(x)$ to be the number m, i.e. $\log_a(x) = m$ is such that $a^m = x$. In this way we can think of logarithms as a different form of writing powers.

Example 3.7. $\log_{10}(1000) = 3$.

The derivative of $\log_a(x)$ is

$$\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x\ln(a)}.$$

Exercise 3.2. Try to show the above statement is true. Hint: use the chain rule.[1]

---

[1] End Lecture 12.

3.3 Exponential growth and decay

Let $y = f(t)$ represent some physical quantity, such as the volume of a substance, the population of a certain species or the mass of a decaying radioactive substance. We want to measure the growth or decay of $f(t)$.

Definition 3.1. *We define the "relative rate of growth (or decay)" as*

$$\frac{\textit{actual rate of growth (or decay)}}{\textit{size of } f(t)} = \frac{f'(t)}{f(t)} = \frac{dy/dt}{y}. \tag{3.1}$$

In many applications, the growth (or decay) rate of the given physical quantity is constant, that is

$$\frac{dy/dt}{y} = \alpha, \quad \text{or} \quad \frac{dy}{dt} = \alpha y, \quad \alpha = \text{constant}.$$

This is a *differential equation* whose solution is

$$y(t) = ce^{\alpha t},$$

where constant c is determined by an initial condition, say, $y(0) = y_0$ (given). Therefore we have

$$y(t) = y_0 e^{\alpha t}.$$

This means that if you start with y_0, after time t you have $y(t)$.

If $\alpha > 0$, the quantity is increasing (growth),

If $\alpha < 0$, the quantity is decreasing (decay).

3.3.1 Radioactive decay

Atoms of elements which have the same number of protons but differing numbers of neutrons are referred to as isotopes of each other. Radioisotopes are isotopes that decompose and in doing so emit harmful particles and/or radiation.

It has been found experimentally that the atomic nuclei of so-called radioactive elements spontaneously decay. They do it at a characteristic rate.

If we start with an amount M_0 of an element with decay rate λ (where $\lambda > 0$), then after time t, the amount remaining is

$$M = M_0 e^{-\lambda t}.$$

This is the radioactive decay equation. The proportion left after time t is

$$\frac{M}{M_0} = e^{-\lambda t},$$

and the proportion decayed is

$$1 - \frac{M}{M_0} = 1 - e^{-\lambda t},$$

3.3.2 Carbon dating

Carbon dating is a technique used by archeologists and others who want to estimate the age of certain artefacts and fossils they uncover. The technique is based on certain properties of the carbon atom.

In its natural state, the nucleus of the carbon atom C^{12} has 6 protons and 6 neutrons. The isotope carbon-14, C^{14}, has 6 protons and 8 neutrons and is radioactive. It decays by beta emission.

Living plants and animals do not distinguish between C^{12} and C^{14}, so at the time of death, the ratio C^{12} to C^{14} in an organism is the same as the ratio in the atmosphere. However, this ratio changes after death, since C^{14} is converted into C^{12} but no further C^{14} is taken in.

Example 3.8. Half-lives: how long before half of what you start with has decayed? When do we get $M = \frac{1}{2}M_0$? We need to solve

$$\frac{M}{M_0} = \frac{1}{2} = e^{-\lambda t},$$

taking logarithms of both sides gives

$$\ln\left(\frac{1}{2}\right) = -\lambda t \qquad \Longrightarrow \qquad t = \frac{\ln(2)}{\lambda}.$$

So, the half-life, $T_{1/2}$ is given by

$$T_{1/2} = \frac{\ln(2)}{\lambda}.$$

If λ is in "per year", then $T_{1/2}$ is in years.

Example 3.9. Carbon-14 (C^{14}) exists in plants and animals, and is used to estimate the age of certain fossils uncovered. It is also used to trade metabolic pathways. C^{14} is radioactive and has a decay rate of $\lambda = 0.000125$ (per year). So we can calculate its half-life as

$$T_{1/2} = \frac{\ln 2}{0.000125} \approx 5545 \text{ years}.$$

Example 3.10. A certain element has $T_{1/2}$ of 10^6 years

1. What is the decay rate?

$$\lambda = \frac{\ln 2}{T_{1/2}} \approx \frac{0.693}{10^3} \approx 7 \times 10^{-7} \text{ (per year)}.$$

2. How much of this will have decayed after 1000 years? The proportion remaining is

$$\frac{M}{M_0} = e^{-\lambda t} = e^{-7\times10^{-7}\times10^3} = e^{-7\times10^{-4}} \approx 0.9993.$$

The proportion decayed is

$$1 - \frac{M}{M_0} \approx 1 - 0.9993 = 0.0007.$$

3. How long before 95% has decayed?

$$\frac{M}{M_0} = 1 - 0.95 = 0.05 = e^{-7\times 10^{-7}t},$$

taking logarithms of both sides we have

$$\ln(0.05) = -7 \times 10^{-7}t$$

which implies

$$t = \frac{\ln(0.05)}{-7 \times 10^{-7}} \approx \frac{-2.996}{-7 \times 10^{-7}} \approx 4.3 \times 10^6 \text{ (years)}.$$

WARNING: Half-life $T_{1/2}$ of a particular element does not mean that in $2 \times T_{1/2}$, the element will completely decay.[2]

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[2] End Lecture 13.

3.3.3 Population growth

Example 3.11. Suppose a certain bacterium divides each hour. Each hour the population doubles:

Hours	1	2	3	4	5	...
Population	2	4	8	16	32	...

After t hours you have 2^t times more bacteria than what you started with. In general, we write it as an exponential form.

If a population, initially P_0 grows exponentially with growth rate λ (where $\lambda > 0$), then at time t, the population is

$$P(t) = P_0 e^{\lambda t}.$$

Example 3.12. Bacterium divides every hour.

1. What is the growth rate?

   We know that when $t = 1$ hour, we are supposed to have

   $$P = 2P_0,$$

   so

   $$2P_0 = P_0 e^{\lambda \cdot 1} \quad \Longrightarrow \quad 2 = e^{\lambda} \quad \Longrightarrow \quad \lambda = \ln 2 \approx 0.693.$$

2. How long for 1 bacterium to become 1 billion?

   $$P_0 = 1, \quad \lambda = 0.693, \quad P = 10^9,$$

   therefore we may write

   $$P = P_0 e^{\lambda t} \quad \Longrightarrow \quad 10^9 = e^{0.693t},$$

   taking logarithms of both sides and re-arranging for t, we have

   $$t = \frac{9 \ln 10}{0.693} \approx 30 \text{ hours}.$$

3.3.4 Interest rate

An annual interest rate of 5% tells you that £100 investment at the start of the year grows to £105. Each subsequent year you leave your investment, it will be multiplied by the factor 1.05.

In general, if you initially invest M_0 (amount) with an annual interest rate r (given as percentage/100), then after t years you have

$$A = M_0(1 + r)^t,$$

where A is the future value. We could write this as an exponential as follows:

$$A(t) = M_0 e^{\lambda t} = M_0(1 + r)^t.$$

Taking logarithms we have

$$\lambda t = t \ln(1+r),$$

so we may write

$$A(t) = M_0 e^{\ln(1+r)t}.$$

ASIDE: In fact, for small r, $\ln(1+r) \approx r$. [3]

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[3] End Lecture 14.

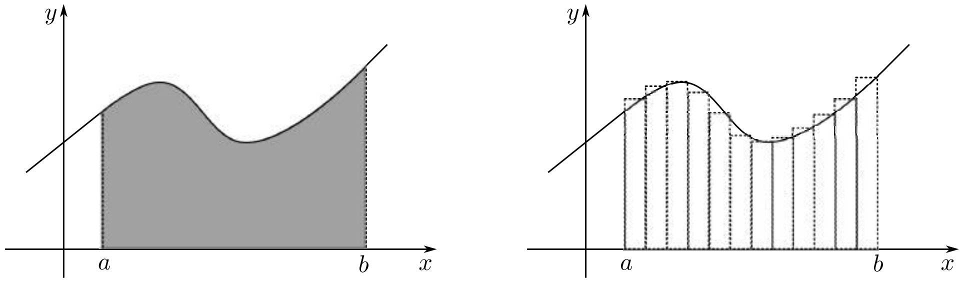
Chapter 4

Integration

4.1 The basic idea

We are interested in calculating areas under curves.

Example 4.1.



(a) *Shaded area under $y = f(x)$.* (b) *Rectangular strips under $y = f(x)$.*

Figure 4.1: *Integration calculates the shaded area under the curve $y = f(x)$, we could do this by dividing the area into rectangles and summing the area of all rectangles.*

How do we do it?

1. We divide the interval $a \leq x \leq b$ into pieces (say equal length).
2. We build a rectangle on each piece, where the top touches the curve.
3. We calculate the total area of the rectangles.

We say, if the division is very fine, we will get a good measure of the area we want. We watch what happens as we make the division of the "strips" finer and finer.

Example 4.2. Consider the function $f(x) = x$ on the interval $0 \leq x \leq 1$.

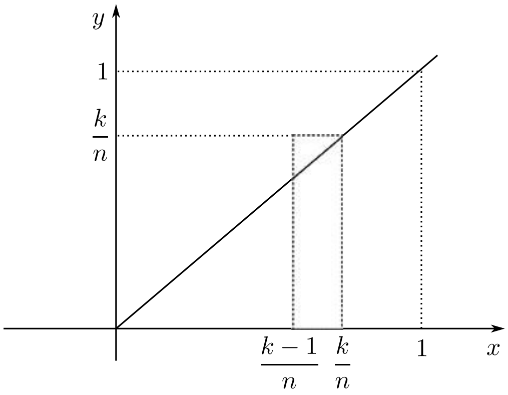


Figure 4.2: *Integrating under the curve $y = x$, from $x = 0$ to $x = 1$.*

We divide $[0, 1]$ into n equal pieces. The divisions occur at

$$0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{k-1}{n}, \frac{k}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1.$$

We have $n + 1$ points and we put a rectangle on each point. The rectangle between $\frac{k-1}{n}$ and $\frac{k}{n}$ will have height $f(\frac{k}{n}) = \frac{k}{n}$ (see Fig. 4.2), and the area of this rectangle is

$$\underbrace{\frac{k}{n}}_{\text{height}} \cdot \underbrace{\frac{1}{n}}_{\text{width}} = \frac{k}{n^2}.$$

The sum of the area of all rectangles on the interval is

$$\begin{aligned} \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{k}{n^2} + \cdots + \frac{n}{n^2} &= \frac{1}{n^2}(1 + 2 + \cdots + k + \cdots + n) \\ &= \frac{1}{n^2}\frac{n(n+1)}{2} \\ &= \frac{1}{2}\left(\frac{n+1}{n}\right) \\ &= \frac{1}{2}\left(1 + \frac{1}{n}\right), \end{aligned}$$

and

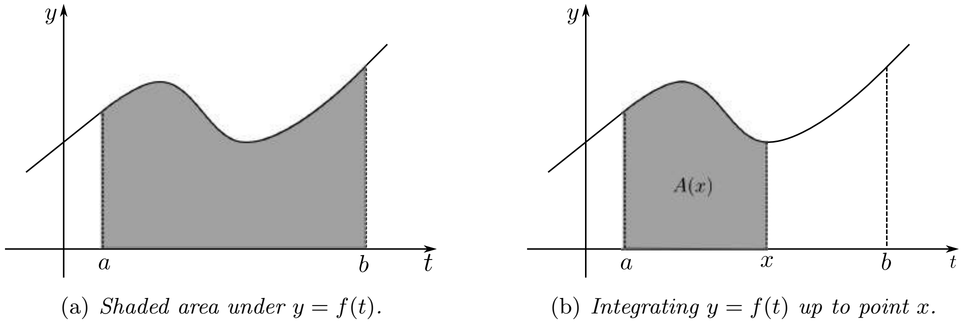
$$\frac{1}{2}\left(1 + \frac{1}{n}\right) \to \frac{1}{2}, \quad \text{as } n \to \infty.$$

That is the sum is approaching the actual area $\frac{1}{2}$.

Therefore, as we increase n so as to get finer divisions, the area approaches the exact area under the curve. You could do this for all functions, however, there is a much quicker way.

Idea: we want to find the area under the curve, let us call it $y = f(t)$.

Example 4.3.



(a) *Shaded area under* $y = f(t)$. (b) *Integrating* $y = f(t)$ *up to point* x.

Figure 4.3: *Writing the shaded area under the curve* $y = f(t)$ *as a function of* x*, where* x *is point within the desired interval* $[a, b]$.

We think of the area as a function of x, say $A(x)$. If we know $A(x)$, then we know the area, i.e. $A(b) - A(a)$. So we want to find this function $A(x)$.

We don't know A, but we can say something about it. Think about how it is related to f. What we know is that

$$A'(x) = f(x). \tag{4.1}$$

Why can we say this? We need to understand what happens to

$$\frac{A(x+h) - A(x)}{h} \quad \text{as } h \to 0. \tag{4.2}$$

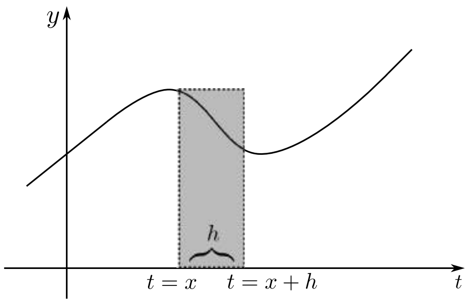


Figure 4.4: *Consider one "strip" under the curve* $y = f(t)$*, to find the limit* (4.2).

The difference $A(x + h) - A(x)$ is the area between $t = x$ and $t = x + h$. So the area is roughly rectangular (if h is small) with height $f(x)$ and base h. So the area is approximately $f(x) \cdot h$. Therefore

$$A(x+h) - A(x) \approx f(x) \cdot h \quad \Longrightarrow \quad \frac{A(x+h) - A(x)}{h} \approx f(x),$$

and

$$\frac{A(x+h)-A(x)}{h} \to f(x) \quad \text{as } h \to 0.$$

Thus, by the definition of the derivative, we have $A'(x) = f(x)$. We defined $A(x)$ as the *antiderivative* of $f(x)$.[1]

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[1] End Lecture 15.

Notice that if $A(x)$ is an antiderivative of $f(x)$, then $A(x) + C$, where C is any constant, is also an antiderivative of $f(x)$.

We employ the following notation:

$$\int f(x)\,dx = F(x) + C, \quad F'(x) = f(x).$$

$\int f(x)\,dx$ is referred to as an *indefinite integral*, where $f(x)$ is called the *integrand*.

4.2 Fundemental Theorem of Calculus

Example 4.4. Consider $f(t) = t^2$. We want to calculate the area under $y = t^2$ between 0 and 1.

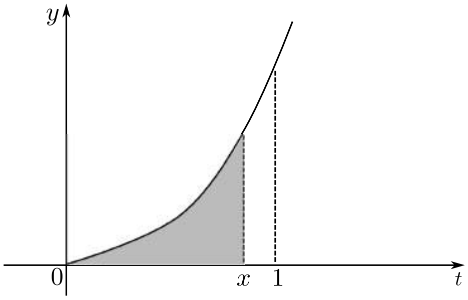

Figure 4.5: *Integrating to find the shaded area under the curve $y = t^2$.*

The area up to x is represented by

$$A(x) = \int_0^x t^2\,dt.$$

We know $A'(x) = x^2 = f(x)$. We need to find $A(x)$. Can we think of a function whose derivative is x^2? Yes! Consider $\frac{1}{3}x^3$. In fact, for any constant C,

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + C\right) = x^2.$$

Our function $A(x)$ is one of the family of functions $\frac{1}{3}x^3 + C$, but which one?

We need to fix the constant C. We have $A(0) = 0$, so if $A(x) = \frac{1}{3}x^3 + C$ then we must have $C = 0$. So finially $A(x) = \frac{1}{3}x^3$. Hence, $A(1) = \frac{1}{3}$, which is the answer to the original question.

We can interpret the above result as

$$\int_0^1 t^2\,dt = A(1) - A(0), \quad A'(x) = x^2.$$

This is actually the fundamental theorem of calculus, which says: If $F'(x) = f(x)$ between a and b, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{4.3}$$

The integral $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ from a (the lower limit) to b (the upper limit).

$\int_a^b f(x)\,dx$ is the limiting value of the sequences derived from the method of divisions when calculating the area under the curve, described previously. Thus, the integral may represent areas under curves.

Example 4.5. Suppose we want to integrate the function x^4 over the interval $[2, 3]$. That is, we want to calculate

$$\int_2^3 x^4\,dx.$$

Think of a function whose derivative is x^4. The answer is $\frac{1}{5}x^5$. So, we write

$$\int_2^3 x^4\,dx = \left[\frac{1}{5}x^5\right]_2^3 = \left[\frac{1}{5}3^5\right] - \left[\frac{1}{5}2^5\right] = \frac{211}{5}.$$

Example 4.6. Suppose we want to integrate the function $1/x^2$ over the interval $[1, 2]$. That is, we want to calculate

$$\int_1^2 \frac{1}{x^2}\,dx.$$

If we put $F(x) = -1/x$, then

$$F'(x) = \frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}.$$

So we can write

$$\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = \left[-\frac{1}{2}\right] - \left[-\frac{1}{1}\right] = \frac{1}{2}.$$

The integral $\int_1^2 \frac{1}{x^2}\,dx$ represents the area under the curve $y = \frac{1}{x^2}$ between 1 and 2, therefore we understand that this integral makes some geometrical sense.

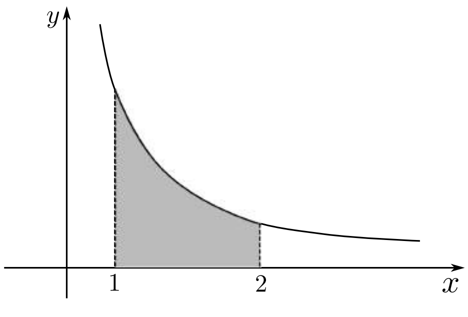

Figure 4.6: *Integrating to find the shaded area under the curve $y = \frac{1}{x^2}$ on the interval* $[1, 2]$.

Example 4.7. Consider the integral of $1/x^2$, but this time on the interval $[-1, 1]$.

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_{-1}^{1} = \left[-\frac{1}{1}\right] - \left[-\frac{1}{-1}\right] = -2.$$

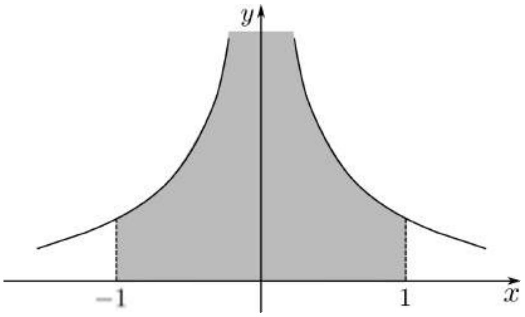

Figure 4.7: *Integrating to find the shaded area under the curve* $y = \frac{1}{x^2}$ *on the interval* $[-1, 1]$*. However, the curve has a vertical asymptote at* $x = 0$*.*

However, the area under the curve in this interval is not −2! What is wrong here? Think about this for next time.[2]

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[2] End Lecture 16.

Previously, we chose an antiderivative which is correct for the given integrand $1/x^2$. However, recall

$$\frac{d}{dx}\left(-\frac{1}{x}\right) \neq \frac{1}{x^2} \quad \text{if } x = 0.$$

That is $F'(x) = f(x)$ doesn't hold for $-1 \leq x \leq 1$. We have to be sure the function is well defined over the entire interval over which we integrate.

Example 4.8. Consider the function $f(x) = x^3$. Then the integral over the interval $[-1, 1]$ is

$$\int_{-1}^{1} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0.$$

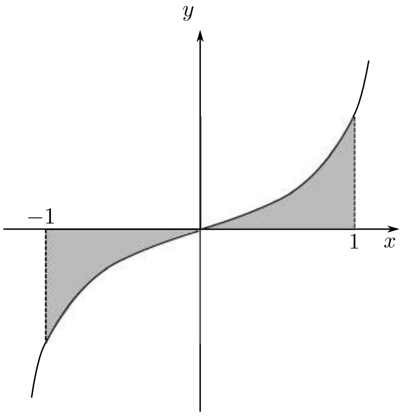

Figure 4.8: *Integrating to find the shaded area under the curve $y = x^3$ on the interval $[-1, 1]$. Have two shaded regions bounded by the curve and the x-axis.*

In this case, the area cancels out. The shaded area is actually given by

$$\int_{0}^{1} x^3\,dx + \left|\int_{-1}^{0} x^3\,dx\right| = \left[\frac{1}{4}x^4\right]_{0}^{1} + \left|\left[\frac{1}{4}x^4\right]_{-1}^{0}\right| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

4.3 Indefinite integrals

So far we know

$$\int f(x)\,dx = F(x) + C, \quad F'(x) = f(x).$$

What function F can we differentiate to get f?

Powers of x:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad (n \neq -1),$$

since

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n.$$

For $n = -1$,

$$\int \frac{1}{x}\,dx = \ln|x| + C,$$

since for $x > 0$

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

and for $x < 0$,

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}.$$

Special rule:

Let us consider the derivative of the logarithm of some general function $f(x)$, i.e.

$$\frac{d}{dx}\left[\ln(f(x))\right] = \frac{1}{f(x)} \cdot \frac{d}{dx}\left[f(x)\right] = \frac{f'(x)}{f(x)}.$$

This implies that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + C,$$

where C is some arbitrary constant of integration. Hence, if we can calculate integrals by inspection if the integrand takes the form $f'(x)/f(x)$ by the above formula.

Example 4.9. Consider the the following integral:

$$I = \int \frac{2x+5}{x^2+5x+3}\,dx.$$

Now, if we choose $f(x) = x^2 + 5x + 3$, then $f'(x) = 2x + 5$. So, if we differentiate $\ln(f(x))$, in this case we have

$$\frac{d}{dx}\left[\ln(x^2+5x+3)\right] = \frac{2x+5}{x^2+5x+3},$$

by the chain rule. Thus, we know the integral must be

$$I = \ln(x^2+5x+3) + C,$$

where C is some arbitrary constant of integration.

Example 4.10. Consider the following integral:

$$I = \int \frac{3}{2x+2}\,dx.$$

Now, if we choose $f(x) = 2x + 2$ then $f'(x) = 2$. However, the numerator of the integrand is 3. Not to worry, as we can simply re-write or manipulate the initial integral as follows:

$$I = \int \frac{3}{2x+2}\,dx = 3\int \frac{1}{2}\frac{2}{2x+2}\,dx = \frac{3}{2}\int \frac{2}{2x+2}\,dx.$$

Since $3/2$ is a constant, which we are able to take out of the integral sign, we need not worry about this and can proceed with the integration using what we have learnt above, giving

$$I = \frac{3}{2}\ln(2x+2) + C.$$

To check, we differentiate the above expression, so

$$\frac{dI}{dx} = \frac{d}{dx}\left[\frac{3}{2}\ln(2x+2) + C\right] = \frac{3}{2}\cdot\frac{1}{2x+2}\cdot 2,$$

which is correct!

This “special case” is an example of a method called *substitution*, and is not limited to integrals which give you logarithms. Nevertheless, it is a good sighter for what’s to follow. Together with inspection, it can be extended for other function by choosing suitable substitutions (i.e. $f(x)$).

Trigonometric functions:

$$\int \cos x\,dx = \sin x + C, \quad \text{since } \frac{d}{dx}(\sin x) = \cos x,$$

$$\int \sin x\,dx = -\cos x + C, \quad \text{since } \frac{d}{dx}(-\cos x) = \sin x.$$

Exponential function:

$$\int e^x\,dx = e^x + C, \quad \text{since } \frac{d}{dx}(e^x) = e^x.$$

As with differentiation, we also have a sum rule for integration, that is

$$\int (f(x) + g(x))\ dx = \int (fx)\,dx + \int g(x)\,dx. \tag{4.4}$$

In other words, the antiderivative of a sum is the sum of the antiderivatives.

Also,

$$\int Kf(x)\,dx = K\int f(x)\,dx, \tag{4.5}$$

where K is a constant, i.e. a constant can be taken outside of the integration sign.

Example 4.11.

$$\begin{aligned}\int (3x^4 + 2x + 5)\,dx &= \int 3x^4\,dx + \int 2x\,dx + \int 5\,dx \\ &= 3\int x^4\,dx + 2\int x\,dx + 5\int\,dx \\ &= \frac{3}{4}x^5 + \frac{2}{3}x^3 + 5x + C.\end{aligned}$$

NOTE: Do not forget the constant C when the integral is indefinite!

Example 4.12.

$$\begin{aligned}\int (x^2-1)(x^4+2)\,dx &= \int x^6 - x^4 + 2x^2 - 2\,dx \\ &= \int x^6\,dx - \int x^4\,dx + 2\int x^2\,dx - \int 2\,dx \\ &= \frac{1}{7}x^7 - \frac{1}{5}x^5 + \frac{2}{3}x^3 - 2x + C.\end{aligned}$$

Example 4.13.

$$\begin{aligned}\int \frac{x^4+1}{x^2}\,dx &= \int\left(\frac{x^4}{x^2}+\frac{1}{x^2}\right)dx\\ &= \int\left(x^2+\frac{1}{x^2}\right)dx\\ &= \int x^2\,dx+\int\frac{1}{x^2}\,dx\\ &= \frac{1}{3}x^3-\frac{1}{x}+C.\end{aligned}$$

4.3.1 Substitution

We can use substitution to convert a complicated integral into a simple one.

Example 4.14. Consider the indefinite integral with integrand $(2x+3)^{100}$. We make the substitution

$$u=2x+3 \quad\Longrightarrow\quad \frac{du}{dx}=2 \quad \text{i.e.} \quad dx=\frac{1}{2}du.$$

So we calculate the integral as follows:

$$\begin{aligned}\int(2x+3)^{100}\,dx &= \int u^{100}\cdot\frac{1}{2}\,du\\ &= \frac{1}{2}\int u^{100}\,du\\ &= \frac{1}{202}(2x+3)^{101}+C.\end{aligned}$$

We can check the result by performing the following differentiation:

$$\frac{d}{dx}\left[\frac{1}{202}(2x+3)^{101}+C\right]=\frac{101}{202}(2x+3)^{100}\cdot 2=(2x+3)^{100},$$

which is correct.

Example 4.15. Suppose we have the integrand $x(x+1)^{50}$. Let us try the substitution

$$u=x+1, \quad \text{so } x=u-1 \quad\Longrightarrow\quad \frac{du}{dx}=1 \quad \text{i.e.} \quad dx=du.$$

So we have

$$\begin{aligned}\int x(x+1)^{50}\,dx &= \int(u-1)u^{50}\,du\\ &= \int u^{51}\,du-\int u^{50}\,du\\ &= \frac{1}{52}u^{52}-\frac{1}{51}u^{51}+C\\ &= \frac{1}{52}(x+1)^{52}-\frac{1}{51}(x+1)^{51}+C\end{aligned}$$

Check:

$$\frac{d}{dx}\left[\frac{1}{52}(x+1)^{52}-\frac{1}{51}(x+1)^{51}+C\right]=(x+1)^{51}-(x+1)^{50}=(x+1)^{50}\cdot x,$$

which is correct.

Example 4.16. Consider the integrand $1/(x\ln x)$. Let us try

$$u = \ln x \qquad \Longrightarrow \qquad \frac{du}{dx} = \frac{1}{x} \quad \text{i.e.} \quad dx = x du.$$

Thus, we calculate the integral as

$$\begin{aligned} \int \frac{1}{x\ln x}\,dx &= \int \frac{1}{xu}\cdot x\,du \\ &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|\ln x| + C. \end{aligned}$$

Check:

$$\frac{d}{dx}\left(\ln|\ln x|\right) = \frac{1}{\ln x}\cdot\frac{1}{x},$$

which is correct.

Example 4.17. Consider the integrand $1/(1+\sqrt{x})$. Let us try the substitution

$$u = 1+\sqrt{x} \qquad \Longrightarrow \qquad \sqrt{x} = u-1.$$

Differentiating we have

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \qquad \Longrightarrow \qquad dx = 2x^{\frac{1}{2}}du = 2(u-1)du.$$

Therefore, we calculate the integral as

$$\begin{aligned} \int \frac{1}{1+\sqrt{x}}\,dx &= \int \frac{1}{u}\cdot 2(u-1)\,du \\ &= 2\int\left(1-\frac{1}{u}\right)\,du \\ &= 2\int du - 2\int\frac{1}{u}\,du \\ &= 2u - 2\ln|u| + C \\ &= 2(1+\sqrt{x}) - 2\ln|1+\sqrt{x}| + C. \end{aligned}$$

Check:

$$\frac{d}{dx}\left[2(1+\sqrt{x}) - 2\ln|1+\sqrt{x}| + C\right] = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}(1+\sqrt{x})} = \frac{1+\sqrt{x}-1}{\sqrt{x}(1+\sqrt{x})} = \frac{1}{1+\sqrt{x}}.$$

Example 4.18. Consider the integrand $\sin(3x+1)$. Let us try the substitution

$$u = 3x+1 \qquad \Longrightarrow \qquad \frac{du}{dx} = 3 \quad \text{i.e.} \quad dx = \frac{1}{3}du.$$

So integrate as follows:

$$\begin{aligned} \int \sin 3x+1\,dx &= \frac{1}{3}\int \sin u\,du \\ &= -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos(3x+1) + C. \end{aligned}$$

Check:

$$\frac{d}{dx}\left[-\frac{1}{3}\cos(3x+1)+C\right] = +\frac{1}{3}\cdot 3\cdot\sin(3x+1) = \sin(3x+1).$$

Example 4.19. Consider the integrand $\sin\left(\frac{1}{x}\right)/x^2$. Let us try the substitution

$$u = \frac{1}{x} \qquad \Longrightarrow \qquad \frac{du}{dx} = -\frac{1}{x^2} \quad \text{i.e.} \quad dx = -x^2 du.$$

So we calculate the integral as follows:

$$\begin{aligned}\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,dx &= \int \frac{\sin u}{x^2} \cdot (-x^2)\,du \\ &= -\int \sin u\,du \\ &= \cos u + C \\ &= \cos\left(\frac{1}{x}\right) + C.\end{aligned}$$

Check:

$$\frac{d}{dx}\left[\cos\left(\frac{1}{x}\right) + C\right] = -\sin\left(\frac{1}{x}\right) \cdot \left(-x^{-2}\right) = \frac{\sin\left(\frac{1}{x}\right)}{x^2},$$

which is correct.

How do we find a suitable substitution? Usually by observation and using our knowledge of differentiation. Or we put

$$u = f(x), \quad \text{then} \quad \frac{du}{dx} = f'(x) \qquad \Longrightarrow \qquad du = f'(x)dx.$$

If we have

$$\frac{d}{dx}[f(x)] = f'(x),$$

then we can write

$$d[f(x)] = f'(x)dx.$$

For instance, in the case

$$\int \frac{1}{x\ln x}\,dx,$$

we know that

$$\frac{d(\ln x)}{dx} = \frac{1}{x} \quad \text{i.e.} \quad d(\ln x) = \frac{1}{x}dx.$$

So we write the integral as

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{x\ln x} \cdot x d(\ln x) = \int \frac{1}{\ln x} d(\ln x).$$

Therefore we know $u = \ln x$ will work!

Similarly we know

$$\frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2}, \quad \text{i.e.} \quad \frac{1}{x^2}dx = -d\left(\frac{1}{x}\right),$$

so the integral from example 4.19 can be written as

$$\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,dx = -\int \sin\left(\frac{1}{x}\right)\,d\left(\frac{1}{x}\right) \qquad \Longrightarrow u = \frac{1}{x}.$$

Also, from example 4.14, we have

$$\int (2x+3)^{100}\,dx = \frac{1}{2}\int (2x+3)^{100}\,d(2x+3),$$

since

$$\frac{d(2x+3)}{dx} = 2 \quad \Longrightarrow \quad dx = \frac{1}{2}d(2x+3) \quad \Longrightarrow \quad \text{let } u = 2x+3,$$

which is the same as what we tried earlier.[3]

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[3] End Lecture 17.

4.3.2 Trigonometric substitution

Example 4.20. We know that

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C,$$

(see pg. 38), since

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Actually, we can work out this integral by a substitution like $x = \sin u$ because we know that

$$1 - \sin^2 u = \cos^2 u,$$

and

$$\frac{dx}{du} = \cos u \quad \text{or} \quad dx = \cos u\, du.$$

Thus, we calculate the integral as

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\, du = \int du = u + C = \sin^{-1} x + C.$$

since

$$\sin^{-1} x = \sin^{-1}(\sin u) = u.$$

Example 4.21.

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C, \quad \text{since} \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Let us try the following

$$x = \tan\theta, \quad \frac{dx}{d\theta} = \frac{1}{\cos^2\theta} = 1 + \tan^2\theta \quad \Longrightarrow \quad dx = (1+\tan^2\theta)d\theta.$$

So we calculate the integral as

$$\int \frac{1}{1+x^2}\, dx = \int \frac{1}{1+\tan^2\theta}(1+\tan^2\theta)\, d\theta = \int d\theta = \theta + C = \tan^{-1} x + C.$$

Since we have

$$x = \tan\theta \quad \Longrightarrow \quad \tan^{-1} x = \tan^{-1}(\tan\theta) = \theta.$$

Example 4.22. Consider the integrand $1/(1+2x^2)$. This is similar to the previous example. If we try

$$\sqrt{2}x = \tan\theta \quad \Longrightarrow \quad dx = \frac{1}{\sqrt{2}}(1+\tan^2\theta)d\theta, \quad \theta = \tan^{-1}(\sqrt{2}x).$$

So we calculate the integral as:

$$\begin{aligned}
\int \frac{1}{1+2x^2}\, dx &= \int \frac{1}{1+(\sqrt{2}x)^2}\, dx \\
&= \frac{1}{\sqrt{2}} \int \frac{1}{1+\tan^2\theta}(1+\tan^2\theta)\, d\theta \\
&= \frac{1}{\sqrt{2}}\theta + C \\
&= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x) + C.
\end{aligned}$$

Ceck:

$$\frac{d}{dx}\left[\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x)+C\right]=\frac{1}{\sqrt{2}}\frac{1}{1+(\sqrt{2}x)^2}\cdot\sqrt{2}=\frac{1}{1+2x^2},$$

which is correct.

Example 4.23. Consider the integrand $x/(1+2x^2)$, which is a variation of the above example. So let us try $\sqrt{2}x=\tan\theta$. Then, we can write

$$\begin{aligned}\int\frac{x}{1+2x^2}\,dx &= \int\frac{\frac{1}{\sqrt{2}}\tan\theta}{1+\tan^2\theta}\cdot\frac{1}{\sqrt{2}}(1+\tan^2\theta\,d\theta\\ &= \frac{1}{2}\int\tan\theta\,d\theta\\ &= \frac{1}{2}\int\frac{\sin\theta}{\cos\theta}\,d\theta\\ &= -\frac{1}{2}\int\frac{1}{\cos\theta}d(\cos\theta).\end{aligned}$$

We achieve the last line from knowing that

$$\frac{d(\cos\theta)}{d\theta}=-\sin\theta.$$

Now we use another substitution to tackle the integral we have so far, that is put $u=\cos\theta$, so

$$\begin{aligned}\int\frac{x}{1+2x^2}\,dx &= -\frac{1}{2}\int\frac{1}{u}\,du\\ &= -\frac{1}{2}\ln|u|+C\\ &= -\frac{1}{2}\ln|\cos\theta|+C\\ &= -\frac{1}{2}\ln\left(\frac{1}{1+\tan^2\theta}\right)^{\frac{1}{2}}+C\\ &= -\frac{1}{4}\ln\left(\frac{1}{1+2x^2}\right)+C\\ &= \frac{1}{4}\ln(1+2x^2)+C.\end{aligned}$$

We gained the final result by using the fact that $|\cos\theta|=\sqrt{\cos^2\theta}$, $\cos^2\theta=1/(1+\tan^2\theta)$ and two log rules.

This result can be gained in a quicker way, that is

$$\begin{aligned}\int\frac{x}{1+2x^2}\,dx &= \frac{1}{4}\int\frac{1}{1+2x^2}\,d(1+2x^2)\\ &= \frac{1}{4}\int\frac{1}{u}\,du\\ &= \frac{1}{4}\ln|u|+C\\ &= \frac{1}{4}\ln(1+2x^2)+C,\end{aligned}$$

where we used the substitution $u=1+2x^2$.

4.3.3 Partial fractions

Example 4.24. Consider the integrand $1/(x^2-1)$. We know that $x^2-1=(x+1)(x-1)$. So we may write

$$\frac{1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1},$$

where A and B are numbers to be found. Re-writing the RHS of the above, we have

$$\frac{A}{x-1}+\frac{B}{x+1}=\frac{(A+B)x+A-B}{(x-1)(x+1)}\equiv\frac{1}{x^2-1}.$$

Therefore, we require

$$\begin{aligned} A+B &= 0, \\ A-B &= 1, \end{aligned}$$

which has solution $A=\frac{1}{2}$ and $B=-\frac{1}{2}$. Therefore the integral becomes

$$\begin{aligned} \int\frac{1}{x^2-1}\,dx &= \frac{1}{2}\int\frac{1}{x-1}\,dx-\frac{1}{2}\int\frac{1}{x+1}\,dx \\ &= \frac{1}{2}\int\frac{1}{x-1}d(x-1)-\frac{1}{2}\int\frac{1}{x+1}d(x+1) \\ &= \frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C. \end{aligned}$$

Example 4.25. Consider the integral

$$\int\frac{x^2+6x+1}{3x^2+5x-2}\,dx.$$

We first factories the denominator so that $3x^2+5x-2=(3x-1)(x+2)$. Also, it would be easier if the polynomial in the numerator was of one degree less than the denominator. So we manipulate the integrand as follows:

$$\begin{aligned} \frac{x^2+6x+1}{3x^2+5x-2} &= \frac{\frac{1}{3}(3x^2+18x+3)}{3x^2+5x-2} \\ &= \frac{\frac{1}{3}(3x^2+5x-2+13x+5)}{3x^2+5x-2} \\ &= \frac{1}{3}\left[1+\frac{13x+5}{3x^2+5x-2}\right]. \end{aligned}$$

Now we proceed by changing the fraction in the square bracket using partial fractions, that is

$$\frac{13x+5}{3x^2+5x-2}=\frac{A}{3x-1}+\frac{B}{x+2}=\frac{(A+3B)x+2A-B}{(3x-1)(x+2)}.$$

Matching the coefficients on the numerator we must have

$$\begin{aligned} A+3B=13 \\ 2A-B=5 \end{aligned}$$

Solving simultaeneously we have the solution $A = 4$ and $B = 3$. So the integral becomes

$$\begin{aligned}
\int \frac{x^2+6x+1}{3x^2+5x-2}\,dx &= \frac{1}{2}\int 1 + \frac{13x+5}{3x^2+5x-2}\,dx \\
&= \frac{1}{3}\int dx + \frac{1}{3}\int \frac{4}{3x-1}\,dx + \frac{1}{3}\int \frac{3}{x+2}\,dx \\
&= \frac{1}{3}\int dx + \frac{4}{9}\int \frac{1}{3x-1}\,d(3x-1) + \frac{3}{3}\int \frac{1}{x+2}\,d(x+2) \\
&= \frac{1}{3}x + \frac{4}{9}\ln|3x-1| + \ln|x+2| + C.
\end{aligned}$$

4.3.4 Integration by parts

This is equivalent to the product rule for integration. Suppose we have two function $u(x)$ and $v(x)$. Then the product rule states

$$\frac{d}{dx}(uv) = u'v + uv'.$$

Rearranging the above gives

$$uv' = \frac{d}{dx}(uv) - u'v.$$

Integrating both sides we get

$$\begin{aligned}
\int uv'\,dx &= \int \frac{d}{dx}(uv)dx - \int u'v\,dx \\
&= uv - \int u'v\,dx.
\end{aligned}$$

So we write the rule for integration by parts as:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx. \tag{4.6}$$

Given an integral whose integrand is the product of two functions, we choose one to be u and the other to be v', from which we can calculate u' and v. Plugging into the above equation hopefully leads to an easier integration.[4]

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[4] End Lecture 18.

Example 4.26. Suppose we have the integrand xe^x. Let us choose

$$u = x, \quad v' = e^x \qquad \Longrightarrow \qquad u' = 1, \quad v = e^x.$$

Therefore, we can calculate the integral as follows:

$$\begin{aligned} \int xe^x \, dx &= \int uv' \, dx \\ &= uv - \int u'v \, dx \\ &= xe^x - \int e^x \, dx \\ &= xe^x - e^x + C. \end{aligned}$$

Check:

$$\frac{d}{dx}\left[xe^x - e^x + C\right] = e^x + xe^x - e^x = xe^x.$$

Example 4.27. Suppose we want to integrate $\ln x = 1 \cdot \ln x$. We choose

$$u = \ln x, \quad v' = 1 \qquad \Longrightarrow \qquad u' = \frac{1}{x}, \quad v = x.$$

So we calculate the integral as

$$\begin{aligned} \int \ln x \, dx &= \int 1 \cdot \ln x \, dx \\ &= \int uv' \, dx \\ &= uv - \int u'v \, dx \\ &= x \ln x - \int \frac{1}{x} \cdot x \, dx \\ &= x \ln x - x + C. \end{aligned}$$

Check:

$$\frac{d}{dx}\left[x \ln x - x + C\right] = \ln x + x \cdot \frac{1}{x} - 1 = \ln x.$$

Example 4.28. Suppose we want to integrate $e^x \cos x$. First let us choose

$$u = \cos x, \quad v' = e^x \qquad \Longrightarrow \qquad u' = -\sin x, \quad v = e^x.$$

So we write our integral as

$$\begin{aligned} \int e^x \cos x \, dx &= \int uv' \, dx \\ &= uv - \int u'v \, dx \\ &= e^x \cos x + \int e^x \sin x \, dx. \end{aligned}$$

Now, we have an integral similar to what we started with, so let us integrate this by parts too, choosing

$$\bar{u} = \sin x, \quad \bar{v}' = e^x \qquad \Longrightarrow \qquad \bar{u}' = \cos x, \quad \bar{v} = e^x.$$

So our original integral becomes

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \cos x + \int \bar{u}\bar{v}' \, dx \\ &= e^x \cos x + \bar{u}\bar{v} - \int \bar{u}'\bar{v} \, dx \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx.\end{aligned}$$

Note, now on the RHS we have the same integral we started with. Rearranging this, we can make the integral the subject, i.e.

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx,$$

$$\therefore \quad 2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x.$$

So finally, we can write

$$\int e^x \cos x \, dx = \frac{1}{2} \left[e^x \left(\cos x + \sin x\right)\right] + C,$$

remembering the constant of integration! Check:

$$\frac{d}{dx}\left[\frac{1}{2}\left[e^x\left(\cos x + \sin x\right)\right] + C\right] = \frac{1}{2}\left\{e^x(\cos x + \sin x) + e^x(-\sin x + \cos x)\right\} = e^x \cos x,$$

which is correct.

4.4 Definite integrals

Remark: all the techniques acquired can be applied to definite integrals.

Example 4.29. Consider the integral

$$I = \int_0^1 (x+1)^3 \, dx.$$

Let us choose the following substitution:

$$u = x + 1 \quad \Longrightarrow \quad dx = du,$$

then the limits of the integral become

$$x = 0 \to u = 1 \quad \text{and} \quad x = 1 \to u = 2.$$

Therefore, we calculate the integral as

$$I = \int_1^2 u^3 \, du = \frac{1}{4} u^4 \bigg|_1^2 = \frac{1}{4}\left[2^4 - 1^4\right] = \frac{15}{4}.$$

This is the same as finding the indefinite integral first,

$$\int (x+1)^3 \, dx = \frac{1}{4}(x+1)^4 + C,$$

then imposing the limits, so

$$I = \int_1^2 u^3 \, du = \left[\frac{1}{4}(x+1)^4 + C\right]\bigg|_{x=1} - \left[\frac{1}{4}(x+1)^4 + C\right]\bigg|_{x=0} = \frac{15}{4}.$$

Example 4.30. Consider the integral

$$I = \int_0^{\frac{\pi}{2}} \cos^3 x \, dx.$$

Let us choose

$$u = \cos^2 x, \quad v' = \cos x \qquad \Longrightarrow \qquad u' = -2\cos x \sin x, \quad v = \sin x.$$

Then the integral is calculated as

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \cos^2 x \cos x \, dx \\ &= \cos^2 x \sin x \,|_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx \\ &= 2\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx. \end{aligned}$$

Now there are two ways to finish the integration.

1.

$$\begin{aligned} I &= 2\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx \\ &= 2\int_0^{\frac{\pi}{2}} \cos x (1 - \cos^2 x) \, dx \\ &= 2\int_0^{\frac{\pi}{2}} \cos x \, dx - 2\int_0^{\frac{\pi}{2}} \cos^3 x \, dx \\ &= 2\sin x|_0^{\frac{\pi}{2}} - 2I \\ &= 2 - 2I. \end{aligned}$$

Finally we can write

$$I = 2 - 2I \qquad \Longrightarrow \qquad I = \frac{2}{3}.$$

2.

$$\begin{aligned} I &= 2\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx \\ &= 2\int_0^{\frac{\pi}{2}} \sin^2 x d(\sin x). \end{aligned}$$

Using the substitution $\bar{u} = \sin x$, then $x = 0 \to u = 0$ and $x = \frac{\pi}{2} \to u = 1$. So, we have

$$I = 2\int_0^1 u^2 \, du = \frac{2}{3}u^3 \Big|_0^1 = \frac{2}{3}.$$

4.5 Numerical integration

Consider evaluating the definite integral

$$\int_a^b f(x)\,dx.$$

In practice, we may only know $f(x)$ at some discrete points, and even if we know $f(x)$, its antiderivative may not be expressed in terms of the functions we know, for example

$$\int \sqrt{1+x^3}\,dx \quad \text{or} \quad \int e^{x^2}\,dx.$$

Since most integrals can not be done analytically, we do them numerically. We do this using a “geometric idea”.

4.5.1 Trapezium method

We want to estimate the integral of $f(x)$ on the interval $[a, b]$, which represents the area under the curve $y = f(x)$ from a to b.

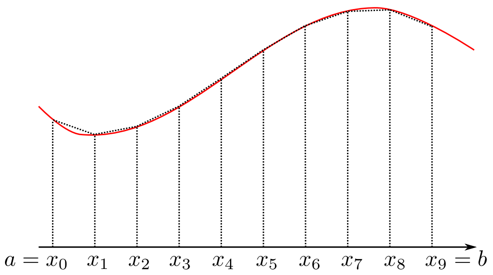

Figure 4.9: *Forming trapeziums with height of the sides dictated by the curve $y = f(x)$ over the interval $[a, b]$.*

We choose n number of pieces. Divide the interval $a \le x \le b$ into n (equal) pieces with points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

On each piece of the interval, we build a trapezium by joining points on the curve by a straight line. We calculate the total area by summing all the area of the trapezia. This is our estimate of the integral.

To start with, let h be the width of one piece of the interval, i.e.

$$h = \frac{b-a}{n},$$

then we have

$$x_k = x_0 + kh, \quad k = 0, 1, 2, \ldots, n. \quad x_0 = a, \quad x_n = b.$$

Let us consider the trapezium based on the piece $[x_{k-1}, x_k]$, whose width is h. The height of the sides of the trapezium are $f(x_{k-1})$ and $f(x_k)$. So the area is

$$h\frac{f(x_{k-1}) + f(x_k)}{2}.$$

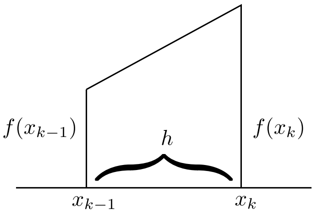

Figure 4.10: *Trapezium constructed over each piece of the interval, where each piece has width h.*

Then the total area under the curve over $[a, b]$ is the sum:

$$\begin{aligned} Area &= h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2} \\ &= \frac{h}{2}\left[(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))\right] \\ &= \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + \cdots + f(x_{n-1})) + f(x_n)\right] \\ &= \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right]. \end{aligned}$$

We can think of the sum as follows, we have the two outer sides of the first and last trapezium, then every trapezium in-between shares its sides with its neighbour, therefore we require two lots of the interior sides.[5]

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[5] End Lecture 19.

Example 4.31. Using the trapezium method, estimate

$$\int_0^1 \frac{1}{1+x^4}\,dx.$$

We choose $n = 4$, then

$$h = \frac{1-0}{4} = \frac{1}{4}, \quad x_k = kh, \quad k = 0, 1, 2, 3, 4.$$

Also, note that

$$f(x) = \frac{1}{1+x^4}, \quad \text{i.e.} \quad f(x_k) = \frac{1}{1+x_k^4}.$$

Therefore, we have

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^4}\,dx &\approx \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3)) + f(x_4)\right] \\
&= \frac{1}{8}\left[f(0) + 2(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)) + f(1)\right] \\
&= \frac{1}{8}\left[1 + 2\left(\frac{256}{257} + \frac{16}{17} + \frac{256}{337}\right) + \frac{1}{2}\right] \\
&= 0.862.
\end{aligned}$$

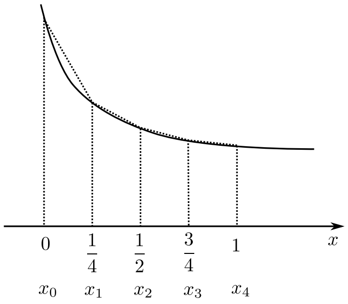


Figure 4.11: *Numerically integrating under $y = 1/(1 + x^4)$. Dividing interval into 4 pieces of width $h = 1/4$.*

This is an over-estimate of the integral since $y = f(x)$ is convex (i.e. it curves up like a cup). If it were concave (i.e. curved down like a cap), then you would have an under-estimate.

Example 4.32. Estimate the following integral

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx, \quad \text{i.e.} \quad f(x) = \frac{1}{\sqrt{1-x^4}}.$$

However, notice that we can't calculate $f(1)$. This is because $y = f(x)$ has a vertical asymptote at $x = 1$.

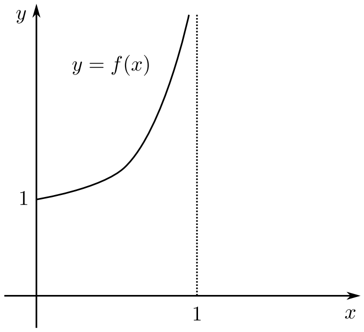


Figure 4.12: *Graph of* $y = 1/(\sqrt{1-x^4})$, *with an asymptote at* $x = 1$. *Estimating integral on interval* $[0, 1]$.

The problem here is that at $x \to 1$, $1 - x^4 \to 0$, rather like $1 - x$.

Since $1 - x^4 = (1-x)(1+x+x^2+x^3)$, i.e. $1 - x^4$ contains a factor $1 - x$, which makes $f(x)$ become *singular* at $x = 1$. So we may try to get rid of it by a substitution.

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx = \int_0^1 \frac{1}{\sqrt{1-x}}\frac{1}{\sqrt{1+x+x^2+x^3}}\,dx.$$

Let us try $u = \sqrt{1-x}$, then $u^2 = 1 - x$ or $x = 1 - u^2$ and $dx = 2 - udu$. Now, at $x = 0 \to u = 1$ and $x = 1 \to u = 0$. Thus, the integral becomes

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx &= \int_1^0 \frac{1}{u}\frac{-2u}{\sqrt{1+1-u^2+(1-u^2)^2+(1-u^2)^3}}\,du \\ &= -2\int_1^0 \frac{1}{\sqrt{4-6u^2+4u^4-u^6}}\,du \\ &= +2\int_0^1 \frac{1}{\sqrt{4-6u^2+4u^4-u^6}}\,du,\end{aligned}$$

where we have applied the rule

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

i.e. if you switch the limits, the integral changes sign.

Let us put

$$g(u) = \frac{1}{\sqrt{4-6u^2+4u^4-u^6}}.$$

Choosing $n = 4$, we have $h = \frac{1}{4}$, so

$$u_k = kh, \quad k = 0, 1, 2, 3, 4.$$

Then, we estimate

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx &= 2\int_0^1 g(u)\,du\\ &\approx 2\cdot\frac{h}{2}\left[g(0)+2\left(g\left(\frac{1}{4}\right)+g\left(\frac{1}{2}\right)+g\left(\frac{3}{4}\right)\right)+g(1)\right]\\ &= 1.32\ldots\end{aligned}$$

The exact result is $1.311\ldots$, so we have a close estimate given we only chose 4 divisions of the interval.

4.6 Application of the definite integral

4.6.1 Area bounded by curves

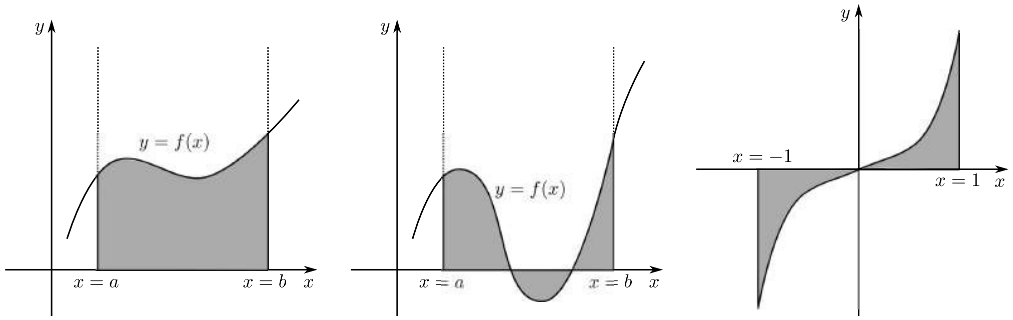
As we have discussed, integrating allows us to find the area bounded by the x-axis and a curve $y = f(x)$. We can extend this to find the area between different curves on the same axis.

If $f(x)$ is a non-negative function on $a \le x \le b$, then $\int_a^b f(x)\,dx$ is the area between the curves $y = f(x)$ and $y = 0$ (i.e. the x-axis) from a to b. That is, the region is bounded by

$$y = f(x), \quad y = 0, \quad x = a, \quad x = b.$$

In general, the area between the curves $y = f(x)$ and x-axis from a to b is

$$\int_a^b |f(x)|\,dx.$$



(a) *Use* $y = f(x)$.   (b) *Use* $y = |f(x)|$.   (c) *Use* $y = |x^3|$.

Figure 4.13: *Examples of area bounded between curves* $y = f(x)$ *and* $y = 0$*, on the interval* $[a, b]$.

Recall:

$$\int_{-1}^{1} x^3\,dx = \left.\frac{x^4}{4}\right|_{-1}^{1},$$

but the shaded area is given by

$$A = \int_{-1}^{1} |x^3|\,dx = \int_{-1}^{0} (-x^3)\,dx + \int_{0}^{1} x^3\,dx = \frac{1}{2}.$$

Suppose the region is bounded above and below by two curves $y = f_1(x)$ (top) and $y = f_2(x)$ (bottom) from a to b, then the area of the region is

$$\int_a^b |f_1(x) - f_2(x)|\,dx. \tag{4.7}$$

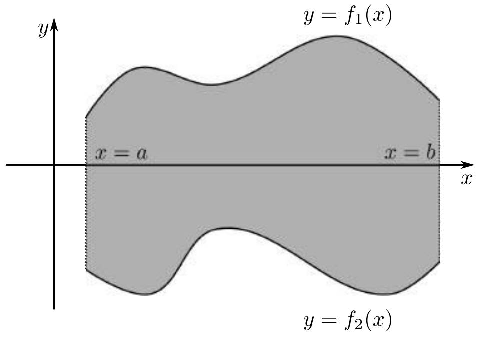


Figure 4.14: *Area bounded between two curves on the interval* $[a, b]$.

Example 4.33. Find the area of the region between $y = x + 1$ and $y = 7 - x$ from $x = 2$ to $x = 5$.

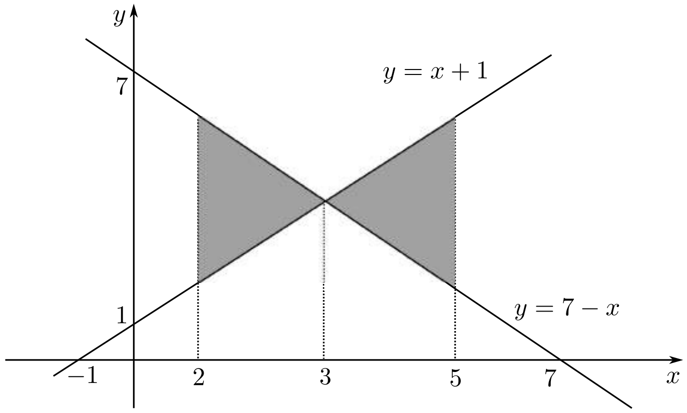


Figure 4.15: *Shaded area between* $y = x + 1$ *and* $y = 7 - x$ *over the interval* $[2, 5]$.

First, we need to find the point where the two curves intersect. The point should satisfy both equations, therefore we solve

$$\left.\begin{array}{l} y = x + 1 \\ y = 7x - 1 \end{array}\right\} \quad \Longrightarrow \quad x + 1 = 7 - x \quad \Longrightarrow \quad \left\{\begin{array}{l} x = 3 \\ y = 4 \end{array}\right.$$

We need to know the point of intersection because from the graph, it is easy to see that on the left of the point of intersection, $y = 7 - x$ is above $y = x + 1$, whilst on the right $y = x + 1$ is above $y = 7 - x$. So we have to be careful when employing the formula (4.7).

Therefore, we can finally calculate the area as follows:

$$\begin{aligned} A &= \int_2^3 [(7 - x) - (x + 1)]\, dx + \int_3^5 [(x + 1) - (7 - x)]\, dx \\ &= \int_2^3 (6 - 2x)\, dx + \int_3^5 (2x - 6)\, dx \\ &= \left[6x - x^2\right]_2^3 + \left[-6x + x^2\right]_3^5 \\ &= 5. \end{aligned}$$

Exercise 4.1. Find the area bounded by the curves $y = x^2$ and $y = 1 - x^2$, on the interval $[-1, 1]$. Hint: is there more than one point of intersection?[6]

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[6] End Lecture 20.

4.6.2 Finding a distance by the integral of velocity

If you know the velocity $v(t)$, then the distance s as a function of time, i.e. $s = s(t)$, is

$$s(t) = \int v(t)\,dt, \quad (\text{since } s'(t) = v(t)).$$

Similarly, if you know the acceleration $a(t)$, then the velocity can be found by the integral of $a(t)$,

$$v(t) = \int a(t)\,dt \quad (\text{since } v'(t) = a(t))$$

Example 4.34. A ball is thrown down from a tall building with an initial velocity of 100ft/sec. Then its velocity after t seconds is given by $v(t) = 32t + 100$. How far does the ball fall between 1 and 3 seconds of elapsed time?

First let us write the distance as the integral of the velocity, that is

$$s(t) = \int v(t) \;= \int 32t + 100\,dt = 16t^2 + 100t + C.$$

Then the distance fallen, $\bar{s}$ say, is given by

$$\bar{s} = s(3) - s(1) = \left.(16t^2 + 100t + C)\right|_{t=3} - \left.(16t^2 + 100t + C)\right|_{t=1} = 328 \text{ ft}.$$

Notice that

$$\begin{aligned} \left.(16t^2 + 100t + C)\right|_{t=3} - \left.(16t^2 + 100t + C)\right|_{t=1} &= \left.(16t^2 + 100t + C)\right|_1^3 \\ &= \int_1^3 (32t + 100)\,dt \\ &= \int_1^3 v(t)\,dt. \end{aligned}$$

In general, if $v(t)$ is a velocity function, then the change in distance between t_1 and t_2 is

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\,dt,$$

and the total distance is

$$s(t) = \int_0^t v(\tau)\,d\tau.$$

NOTE: here we call τ a *dummy variable.*

Chapter 5

Differential Equations

Differential equations are equations involving a certain unknown function and its derivative (equations for functions). For example

$$y' = x^2 \quad \text{or} \quad \frac{dy}{dx} = x^2,$$

where y is a function of x, i.e. $y = y(x)$. We know that in this particular case,

$$y(x) = \frac{1}{3}x^3 + C,$$

where C is an arbitrary constant.

Recall exponential growth and decay: we define the relative rate of growth as y'/y. If we assume that the rate is constant, say λ, then we have

$$y' = \lambda y, \quad (y = y(t)). \tag{5.1}$$

This has solution $y = Ae^{\lambda t}$, since

$$y'(t) = \lambda A e^{\lambda t} = \lambda y(t),$$

where A is an unknown constant.

The solution $Ae^{\lambda t}$ is called the *general solution* to the differential equation in (5.1), implying that we have found all possible solutions.

To determine the constant, we need an extra piece of information. For instance, if we know y at the initial time, say $y(0) = y_0$ (given), then

$$y(0) = Ae^{\lambda \cdot 0} = A = y_0,$$

so $y(t) = y_0 e^{\lambda t}$.

The problem $y' = \lambda y$, $y(0) = y_0$ is call an *initial value problem* (IVP), which has a unique solution $y = y_0 e^{\lambda t}$.

The *order* of a differential equation is the order of the highest derivative appearing in the equation.

Example 5.1.

$y' + 2xy = e^x$ - first order,

$y'' + 3y' + 4y = 0$ - second order,

$(y')^2 + 2\ln y + 4e^x = x^3$ - first order,

$y^{(3)} + 3y'' + 2y' + 4y = 10$ - third order.

5.1 First order differential equations

Here we will consider different techniques to solve first order ordinary differential equations (1st order ODEs).

5.1.1 Separation of variables

Let us consider the simplest case of a first order differential equation, that is

$$\frac{dy}{dx} = f(x, y), \tag{5.2}$$

for example

$$\begin{aligned} &y' + 2xy = e^x \quad \Longrightarrow \quad y' = -2xy + e^x = f(x, y),\\ &(y')^2 + 2\ln y + 4e^x = x^3 \quad \Longrightarrow \quad y' = \pm\sqrt{x^3 - 2\ln y + 4e^x} = f(x, y),\\ &y^2 y' - x = 0 \quad \Longrightarrow \quad y' = x/y^2 = f(x, y). \end{aligned}$$

Suppose that $f(x, y)$ is separable, i.e. it can be written as

$$f(x, y) = g(x)h(y), \tag{5.3}$$

for example

$$f(x, y) = \frac{x}{y^2} = x \cdot \frac{1}{y^2} \quad \Longrightarrow \quad g(x) = x, \quad h(y) = \frac{1}{y^2}.$$

Then we have

$$\frac{1}{h(y)}\, dy = g(x)\, dx,$$

where the LHS only depends on y and the RHS only depends on x. Integrating both sides we have

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx.$$

Therefore, if we can work out the integrals, then we can obtain the general solution to the equation. Also, since we expect a constant of integration, we can merge the constants from both sides into one constant, say C, since we are integrating the equation once only.[1]

---

[1] End Lecture 21.

Example 5.2. Consider the differential equation

$$y' = \lambda y.$$

We already know the solution to this equation. Now let us see how to derive it using separation of variables.

$$\frac{dy}{dt} = \lambda y,$$

taking all things relating to y to the left, and for t to the right, we have

$$\frac{1}{y} dy = \lambda dt.$$

Integrating both sides we have

$$\int \frac{1}{y}\, dy = \int \lambda\, dt,$$

hence, using what we have learnt in previous chapters we get

$$\ln y = \lambda t + C.$$

Finally, re-arranging for y, we have

$$y = e^{\lambda t + C} = Ae^{\lambda t}, \quad A = e^C.$$

Example 5.3. Consider the equation

$$\frac{dy}{dx} = xy,$$

following the procedure as in the previous example, we have

$$\frac{1}{y}\, dy = x\, dx.$$

Integrating both sides we have

$$\begin{aligned} & \int \frac{1}{y}\, dy = \int x\, dx, \\ \Longrightarrow \qquad & \ln y = \frac{1}{2}x^2 + C. \end{aligned}$$

Taking exponentials of both sides in order to re-arrange for y, we get

$$y = e^{\frac{1}{2}x^2 + C} = Ae^{\frac{1}{2}x^2}, \quad A = e^C.$$

We can check if this satisfies the original equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(Ae^{\frac{1}{2}x^2}\right) = Ae^{\frac{1}{2}x^2+} \cdot \frac{1}{2} \cdot 2x = xy.$$

Example 5.4. Consider the differential equation

$$y^2 y' = x.$$

We first write it in the form $y' = f(x, y)$, i.e.

$$\frac{dy}{dx} = \frac{x}{y^2},$$

now we realises that we can apply separation of variable, so

$$
\begin{aligned}
& & y^2 dy &= x\,dx, \\
&\implies & \int y^2 dy &= \int x\,dx \\
&\implies & \frac{1}{3}y^3 &= \frac{1}{2}x^2 + C, \\
&\implies & y &= \left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}},
\end{aligned}
$$

where C' is some constant (different to C, since we multiplied through by 3). Again, we check the solution satisfies the equation

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}\right] \\
&= \frac{1}{3}\left(\frac{3}{2}x^2 + C'\right)^{-\frac{2}{3}} \cdot \frac{3}{2} \cdot 2x \\
&= x\left(\frac{3}{2}x^2 + C'\right)^{-\frac{2}{3}} \\
&= x\left[\left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}\right]^{-2} \\
&= \frac{x}{y^2}.
\end{aligned}
$$

Example 5.5. Consider the following initial-value problem:

$$\frac{dy}{dx} = y^2(1 + x^2), \quad y(0) = 1.$$

First, we find the general solution, note, we can use separation of variables in this example, so

$$
\begin{aligned}
& & \frac{1}{y^2}\,dy &= (1 + x^2)\,dx \\
&\implies & \int \frac{1}{y^2}\,dy &= \int (1 + x^2)\,dx \\
&\implies & -\frac{1}{y} &= x + \frac{1}{3}x^3 + C \\
&\implies & y &= -\frac{1}{x + \frac{1}{3}x^3 + C}.
\end{aligned}
$$

Now check that the general solution satisfies the original differential equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\frac{1}{x + \frac{1}{3}x^3 + C}\right) = \frac{1 + x^2}{\left(x + \frac{1}{3}x^3 + C\right)^2} = (1 + x^2)y^2.$$

Now it remains to find the constant C, by applying the condition $y(0) = 1$, i.e. we put $x = 0$.

$$y(0) = -\frac{1}{0 + \frac{1}{3} \cdot 0^3 + C} = -\frac{1}{C} = 1, \quad \implies \quad C = -1.$$

So the solution to the initial value problem is

$$y = \frac{1}{1 - x - \frac{1}{3}x^3}.$$

Example 5.6. Consider the initial value problem

$$e^y y' = 3x^2, \quad y(0) = 2.$$

First, find the general solution,

$$\begin{aligned} & e^y y' = 3x^2 \\ \Longrightarrow \quad & \frac{dy}{dx} = 3x^2 e^{-y} \\ \Longrightarrow \quad & e^y\, dy = 3x^2\, dx \\ \Longrightarrow \quad & \int e^y\, dy = \int 3x^2\, dx \\ \Longrightarrow \quad & e^y = x^3 + C \\ \Longrightarrow \quad & y = \ln(x^3 + C). \end{aligned}$$

Check:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(x^3 + C)\right) = \frac{3x^2}{x^3 + C} = 3x^2 \frac{1}{x^3 + C}.$$

Recall $e^{\ln(a)} = a$, using this, we can write

$$\frac{dy}{dx} = 3x^2 e^{\ln\left(\frac{1}{x^3+C}\right)} = 3x^2 e^{-\ln(x^3+C)} = 3x^2 e^{-y}.$$

Now we apply the initial condition,

$$y(0) = \ln(C) = 2 \qquad \Longrightarrow \qquad C = e^2,$$

so we have the final solution

$$y(x) = \ln(x^3 + e^2).$$

5.1.2 Linear first-order differential equations

An n-th order differential equation is linear if it can be written in the form:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + a_{n-2}(x)y^{(n-2)} + \cdots + a_1(x)y' + a_0(x)y = f(x), \tag{5.4}$$

or

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{5.5}$$

where a_i $(i = 0, 1, 2, \ldots, n)$ and $f(x)$ are known functions of x.

NOTE: here we have leading order coefficient of 1.

NOTATION: the n-th derivative of y with respect to x is written as

$$y^{(n)} \equiv \frac{d^n y}{dx^n}.$$

Example 5.7.

1. $$y' + 2y = e^x,$$
   is a first-order linear differential equation with constant coefficients, i.e.
   $$n = 1, \quad a_0(x) = 2, \quad f(x) = e^x.$$

2. $$y'' + 8y' + 16y = 0,$$
   is a second-order linear differential equation with constant coefficients, i.e.
   $$n = 2, \quad a_1(x) = 8, \quad a_0(x) = 16, \quad f(x) = 0.$$

3. $$y'' + yy' = x,$$
   is a second-order *non-linear* differential equation.

4. $$e^x y' + xy = \cos x \iff y' + xe^{-x} y = e^{-x}\cos x,$$
   is a first-order linear differential equation with
   $$n = 1, \quad a_0(x) = xe^{-x}, \quad f(x) = e^{-x}\cos x.$$

5. $$\frac{dx}{dt} = 3x + t^3 e^{3t},$$
   is a first-order linear differential equation with
   $$n = 1, \quad a_0(t) = 3, \quad f(t) = t^3 e^{3t}.$$

Definition 5.1. *If $f(x) \equiv 0$ (zero function), then the linear equation* (5.5) *is said to be homogeneous; otherwise, we say the equation is non-homogenous.*

Example 5.8.

$$\begin{aligned} y' &+2y = e^x && \text{- non-homogeneous,} \\ y' &+2y = 0 && \text{- homogeneous.} \end{aligned}$$

A first-order linear equation has the form

$$\frac{dy}{dx} + q(x)y = p(x). \tag{5.6}$$

If $p(x) \equiv 0$, then we can try to solve the differential equation by separation of variables.

In general, we solve as follows. Let

$$T(x) = \int q(x)\,dx,$$

then

$$e^{T(x)} = e^{\int q(x)\,dx},$$

is called the *integrating factor*. We will use this integrating factor to derive the general solution to the first-order linear differential equation.[2]

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[2] End Lecture 22.

Consider the general case for a first-order differential equation given by (5.6). First let us multiply both sides of the equation by $e^{T(x)}$,

$$e^{T(x)}\frac{dy}{dx} + e^{T(x)}q(x)y = e^{T(x)}p(x). \tag{5.7}$$

Now let us consider the derivative of $e^{T(x)}y$,

$$\begin{aligned}
\frac{d}{dx}(e^{T(x)}y) &= e^{T(x)}\frac{dy}{dx} + y\frac{d}{dx}(e^{T(x)}) \\
&= e^{T(x)}\frac{dy}{dx} + ye^{T(x)}\frac{d}{dx}(T(x)) \\
&= e^{T(x)}\frac{dy}{dx} + ye^{T(x)}q(x) \\
&= e^{T(x)}p(x).
\end{aligned}$$

Here we have put $T'(x) = q(x)$ and applied (5.7). So we have

$$\frac{d}{dx}(e^{T(x)}y) = e^{T(x)}p(x).$$

Therefore, integrating both sides we have

$$e^{T(x)}y = \int e^{T(x)}p(x)\,dx,$$

or

$$y = e^{-T(x)}\int e^{T(x)}p(x)\,dx.$$

we have shown that if $y' + q(x)y = p(x)$ and $T(x) = \int q(x)\,dx$, then

$$y = e^{-T(x)}\int e^{T(x)}p(x)\,dx. \tag{5.8}$$

Note, we expect a constant when we complete the integration above.

Example 5.9. Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x,$$

note that $f(x,y) = x - (y/x)$ can't be separated. So we put

$$q(x) = \frac{1}{x}, \quad p(x) = x \qquad \Longrightarrow \qquad T(x) = \int q(x)\,dx = \int \frac{1}{x}\,dx = \ln x.$$

Then we have the solution

$$y = e^{-\ln x}\int e^{\ln x}x\,dx = e^{\ln\frac{1}{x}}\int x^2\,dx = \frac{1}{x}\left[\frac{1}{3}x^3 + C\right] = \frac{1}{3}x^2 + \frac{C}{x}.$$

Example 5.10. Consider

$$\frac{dy}{dx} + xy = x.$$

So we put

$$q(x) = x, \quad p(x) = x \qquad \Longrightarrow \qquad T(x) = \int q(x)\,dx = \int x\,dx = \frac{1}{2}x^2.$$

Then the solution is

$$y = e^{-\frac{1}{2}x^2} \int e^{\frac{1}{2}x^2} x\,dx = e^{-\frac{1}{2}x^2} \left[\int e^{\frac{1}{2}x^2}\, d\left(\frac{1}{2}x^2\right) \right] = e^{-\frac{1}{2}x^2} \left[e^{\frac{1}{2}x^2} + C \right],$$

i.e.

$$y = 1 + Ce^{-\frac{1}{2}x^2}.$$

NOTE: this example could have been done using separation of variables.

Example 5.11. Solve the initial-value problem:

$$y' = y + x^2, \quad y(0) = 1,$$

so we put

$$q(x) = -1, \quad p(x) = x^2, \quad T(x) = \int -1\,dx = -x.$$

Therefore, the solution is

$$y = e^x \int e^{-x} x^2\,dx.$$

We calculate the integral using integration by parts,

$$\begin{aligned} \int e^{-x} x^2\,dx &= -x^2 e^{-x} + \int 2xe^{-x}\,dx \\ &= -x^2 e^{-x} - 2xe^{-x} + \int 2e^{-x}\,dx \\ &= e^{-x}\left[-x^2 - 2x\right] - 2\int e^{-x} \\ &= -e^{-x}\left(x^2 + 2x + 2\right) + C. \end{aligned}$$

Hence, the solution to the differential equation is

$$y = -(x^2 + 2x + 2) + Ce^x.$$

It remains to use the initial condition to find C, i.e.

$$y(0) = -2 + Ce^0 = 1 \quad \Longrightarrow \quad C = 3,$$

so the final solution is

$$y = -(x^2 + 2x + 2) + 3e^x.$$

In the previous three examples, we gained the following results:

Ex. 5.9.

$$y' + \frac{1}{x}y = x, \quad y = \frac{1}{3}x^2 + C \cdot .\frac{1}{x},$$

Ex. 5.10.

$$y' + xy = x, \quad y = 1 + C \cdot e^{-\frac{1}{2}x^2},$$

Ex. 5.11.

$$y' - y = x^2, \quad y = -(x^2 + 2x + 2) + Ce^x.$$

These examples have something very important in common, that is the solutions have the following form

$$y = f(x) + Cg(x),$$

with explicit functions f and g. Here $y = f(x)$ is a particular solution (take $C = 0$) of the non-homogeneous equation, and $y = g(x)$ is a solution of the corresponding homogeneous equation. For example

Ex. 5.9.

$$\begin{aligned} &\text{if} \quad y = \frac{1}{3}x^2, \quad \text{then } y' + \frac{1}{x}y = \frac{2}{3}x + \frac{1}{x} \cdot \frac{1}{3}x^2 = x, \\ &\text{if} \quad y = \frac{1}{x}, \quad \text{then } y' + \frac{1}{x}y = -\frac{1}{x^2} + \frac{1}{x} \cdot \frac{1}{x} = 0. \end{aligned}$$

Ex. 5.10.

$$\begin{aligned} &\text{if} \quad y = 1, \quad \text{then } y' + xy = 0 + x = x, \\ &\text{if} \quad y = e^{-\frac{1}{2}x^2}, \quad \text{then } y' + xy = e^{-\frac{1}{2}x^2}\left(\frac{1}{2} \cdot 2x\right) + xe^{-\frac{1}{2}x^2} = 0. \end{aligned}$$

These examples reveal an intrinsic structure of the general solution of a linear differential equation. We solve the equation by finding the solution of its homogeneous equation, and a particular solution to the non-homogeneous equation.

Now we will understand how to use this method to solve a first-order linear differential equation with constant coefficients:

$$y' + \lambda y = p(x), \quad \lambda \text{ is constant.}$$

We know that the general solution is

$$y(x) = \underbrace{f(x)}_{\text{particular integral (P.I.)}} + \underbrace{Cg(x)}_{\text{complementary function (C.F.)}},$$

and

$$f' + \lambda f = p(x), \quad g' + \lambda g = 0,$$

where C is the constant of integration to be found. We start by building g. So we need to solve

$$g' = \lambda g = 0,$$

solution:

$$g = Ce^{-\lambda x}.$$

Therefore, the general to $y' + \lambda y = p(x)$ is

$$y = f(x) + Ce^{-\lambda x},$$

where f is a particular solution (depending on $p(x)$). In what follows, we shall find $f(x)$ for certain kinds of function $p(x)$.[3]

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[3] End Lecture 23.

We will consider three different kinds of options for $p(x)$:

1. Polynomial,
2. trigonometric functions sin or cos,
3. exponential function:
   (a) p has the form e^{rx} where $r \neq -\lambda$,
   (b) p has the form e^{rx} where $r = -\lambda$.

Example 5.12. Consider the differential equation

$$y' + y = x,$$

so we have

$$\lambda = 1, \quad p(x) = x, \quad \text{solution: } y = f(x) + Ce^{-x},$$

i.e. we see that the C.F. from the homogeneous equation is $g = Ce^{-x}$. Now, here f should be a polynomial with a degree of one, since $p(x) = x$. So we try the most general first order polynomial, $f(x) = ax + b$, and so $f'(x) = a$. Substituting $y = f(x)$ into the differential equation we have that

$$f' + f = a + ax + b = ax + (a + b) \equiv x,$$

Thus, comparing coefficients from the LHS and RHS we must have that

$$\left.\begin{array}{r} a = 1 \\ a + b = 0 \end{array}\right\} \implies \left.\begin{array}{r} a = 1 \\ b = -1 \end{array}\right\} \implies f(x) = x - 1,$$

so $f(x) = x - 1$ is a solution. Therefore, the general solution to the original equation is

$$y(x) = x - 1 + Ce^{-x}.$$

Example 5.13. Consider the differential equation

$$y' + 2y = x^2, \quad \lambda = 2, \quad p(x) = x^2.$$

We can easily work out the C.F., the general solution will take the form

$$y(x) = f(x) + Ce^{-2x}.$$

For the particular integral, we try $f(x) = ax^2 + bx + c$, since $p(x) = x^2$. So, we have $f'(x) = 2ax + b$. Substituting into the differential equation, we have

$$f' + 2f = 2ax + b + 2ax^2 + 2bx + 2c = 2ax^2 + (2a + 2b)x + (b + 2c) \equiv ex^2.$$

Comparing coefficients between the LHS and RHS we have

$$\left.\begin{array}{r} 2a = 1 \\ a + b = 0 \\ b + 2c = 0 \end{array}\right\} \implies \left.\begin{array}{r} a = \frac{1}{2} \\ b = -\frac{1}{2} \\ c = \frac{1}{4} \end{array}\right\} \implies f(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4},$$

and the general solution is

$$y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}.$$

Example 5.14. Consider the differential equation

$$y' + y = \sin 2x, \quad \lambda = 1, \quad p(x) = \sin 2x.$$

The general solution will have the form

$$y(x) = f(x) + Ce^{-x}.$$

To find the particular integral, we try $f(x) = a\sin 2x + b\cos 2x$, since $p(x) = \sin 2x$. Therefore $f'(x) = 2a\cos 2x - 2b\sin 2x$. Substituting into the differential equation tells us

$$f' + f = 2a\cos 2x - 2b\sin 2x + a\sin 2x + b\cos 2x = (a - 2b)\sin 2x + (2a + b)\cos 2x \equiv \sin 2x.$$

Comparing coefficient from LHS and RHS then

$$\left.\begin{array}{r} a - 2b = 1 \\ 2a + b = 0 \end{array}\right\} \implies \left.\begin{array}{r} a = \frac{1}{5} \\ b = -\frac{2}{5} \end{array}\right\} \implies f(x) = \frac{1}{5}\sin 2x - \frac{2}{5}\cos 2x.$$

So the general solution is

$$y(x) = \frac{1}{5}\sin 2x - \frac{2}{5}\cos 2x + Ce^{-x}.$$

Example 5.15. Consider the differential equation

$$y' + y = e^{2x}, \quad \lambda = 1, \quad p(x) = e^{2x}, \quad r = 2 \neq -\lambda = -1.$$

The general solution takes the form

$$y(x) = f(x) + Ce^{-x}.$$

For the particular integral, we shall try $f(x) = ae^{2x}$, since $p(x) = e^{2x}$ and $f'(x) = 2ae^{2x}$. Substituting into the differential equation gives

$$f' + f = 2ae^{2x} + ae^{2x} = 3ae^{2x} \equiv e^{2x}.$$

Comparing coefficients gives $a = \frac{1}{3}$, thus the general solution is

$$y(x) = \frac{1}{3}e^{2x} + Ce^{-x}.$$

Example 5.16. Consider the differential equation

$$y' + y = e^{-x}, \quad \lambda = 1, \quad p(x) = e^{-x}.$$

The general solution has the form

$$y(x) = f(x) + Ce^{-x}.$$

Now, if we try $f(x) = ae^{-x}$, then

$$f' + f = -ae^{-x} + ae^{-x} = 0 \neq e^{-x},$$

so it doesn't work since $r = -1 = -\lambda$. But we may try

$$f(x) = axe^{-x}, \quad \text{then } f'(x) = ae^{-x} - axe^{-x}.$$

Substituting into the differential equation gives

$$f'(x) + f(x) = -axe^{-x} + ae^{-x} + axe^{-x} = ae^{-x} \equiv e^{-x},$$

therefore, comparing coefficients, we have $a = 1$, so the general solution is

$$y(x) = xe^{-x} + Ce^{-x} = e^{-x}(x + C).$$

In general, for

$$y' + \lambda y = e^{-\lambda x},$$

the general solution is of the form

$$y(x) = f(x) + Ce^{-\lambda x}.$$

For the particular integral we try $f(x) = axe^{-\lambda x}$, then

$$f' + \lambda f = ae^{-\lambda x} - \lambda axe^{-\lambda x} + \lambda axe^{-\lambda x} = ae^{-\lambda x} \equiv e^{-\lambda x},$$

therefore, comparing coefficient, we have $a = 1$. So the general solution is

$$y(x) = xe^{-\lambda x} + Ce^{-\lambda x} = e^{-\lambda x}(x + C).$$

This can be easily generalised for the case

$$y' + \lambda y = be^{-\lambda x}, \quad \lambda \text{ and } b \text{ are constants.}$$

5.1.3 Second-order linear differential equations with constant coefficients

$$y'' + ry' + sy = p(x), \tag{5.9}$$

where r and s are constant. The general solution to (5.9) has the form

$$y(x) = f(x) + g(x),$$

where $y = f(x)$ is a *particular integral* from the solution to (5.9) and $y = g(x)$ is the *complementary function* from the solution to the homogeneous equation

$$y'' + ry' + sy = 0. \tag{5.10}$$

Solving (5.10) to build $y = g(x)$

Two functions (say, two solutions to (5.10)) are said to be independent if one is not a *constant* multiple of the other; otherwise, they are said to be dependent. For example

- $y_1(x) = 1$, $y_2(x) = x$, here y_1 and y_2 are independent.
- $y_1(x) = e^{ax}$, $y_2(x) = e^{bx}$, if $a \neq b$, then y_1 and y_2 are independent since

  $$y_1 = e^{ax} = e^{bx+(a-b)x} = e^{(b-a)x}y_2,$$

  i.e. if $a \neq b$, then the multiple is not a constant.
- $y_1(x) = e^{ax}$, $y_2(x) = e^{ax+b}$, a and b are constant, then y_1 and y_2 are dependent, since $y_2(x) = e^b y_1(x)$, e^b is constant.[4]

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[4] End Lecture 24.

If $y_1(x)$ and $y_2(x)$ are independent and they are the solutions to (5.10) then the general solution to (5.10) can be written in the form

$$y(x) = C_1 y_1(x) + C_2 y_2(x), \tag{5.11}$$

where C_1 and C_2 are constants. In other words, if you have two independent solutions, then you can represent any other solution in terms of these two solutions.

Now, we verify that (5.11) is the solution to (5.10):

$$\begin{aligned} y'' + ry' + sy &= C_1 y_1'' + C_1 r y_1' + C_1 s y_1 + C_2 y_2'' + C_2 r y_2' + C_2 s y_2 \\ &= C_1 \underbrace{(y_1'' + r y_1' + s y_1)}_{=0} + C_2 \underbrace{(y_2'' + r y_2' + s y_2)}_{=0} \\ &= 0. \end{aligned}$$

Therefore we need to find two independent solutions to (5.10). But how do we find them? Let us assume that the solutions are of the form

$$y = e^{\lambda x}.$$

Then

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x},$$

and so substituting into (5.10) we have

$$y'' + ry' + sy = \lambda^2 e^{\lambda x} + r\lambda e^{\lambda x} + s e^{\lambda x} = e^{\lambda x}(\lambda^2 + r\lambda + s) \equiv 0,$$

which tells us that if λ is a root of the equation

$$\lambda^2 + r\lambda + s = 0, \tag{5.12}$$

then $y = e^{\lambda x}$ will be a solution to (5.10). Equation (5.12) is called the *auxiliary equation* or *characteristic equation* and is quadratic, so we expect two solutions for λ and thus two independent solutions to (5.10).

Example 5.17. Consider the second-order differential equation

$$y'' + 3y' + 2y = 0.$$

If we put $y = e^{\lambda x}$, we see that the auxiliary equation is

$$\lambda^2 + 3\lambda + 2 = 0 \iff (\lambda + 1)(\lambda + 2) = 0.$$

This has two roots:

$$\lambda_1 = -1, \quad \lambda_2 = -2.$$

Therefore we have two solutions e^{-x} and e^{-2x}, and they are independent. So the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{-2x}.$$

In general, we know that the roots of (5.12) are given by

$$\lambda = \frac{-r \pm \sqrt{\Delta}}{2}, \quad \Delta = r^2 - 4s.$$

There are three cases, depending on the value of Δ.

(a) $\Delta > 0$: two distinct real roots (see Fig. 5.1(a)),

$$\lambda_1 = \frac{-r+\sqrt{\Delta}}{2}, \quad \lambda_2 = \frac{-r-\sqrt{\Delta}}{2}.$$

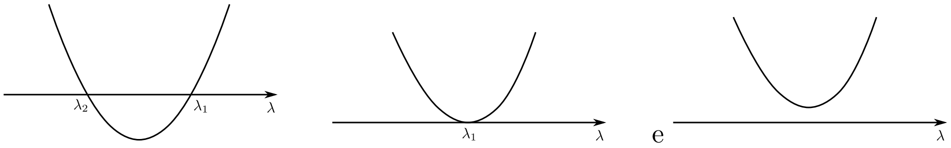
We have two independent solutions to the differential equation (5.10),

$$e^{\lambda_1 x} \quad \text{and} \quad e^{\lambda_2 x},$$

so

$$g(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x},$$

is the general solution to (5.10), i.e. the C.F. of (5.9).



(a) *Two roots.* (b) *One roots.* (c) *No roots (complex).*

Figure 5.1: *Different options for the curve* $y = \lambda^2 + r\lambda + s$ *when solving the equation* $\lambda^2 + r\lambda + s = 0$.

(b) $\Delta = 0$, (5.12) has one root which is real (see Fig. 5.1(b)), given by

$$\lambda_1 = \frac{r}{s}, \quad \Delta = 0 \quad \Longrightarrow \quad s = \frac{r^2}{4}.$$

So $e^{\lambda_1 x}$ is one solution to (5.10). We need to find another solution, which is independent of $e^{\lambda_1 x}$. So we try

$$y = xe^{\lambda_1 x},$$

then

$$y' = e^{\lambda_1 x} + \lambda_1 x e^{\lambda_1 x} \quad \text{and} \quad y'' = \lambda_1 e^{\lambda_1 x} + \lambda_1 e^{\lambda_1 x} + \lambda_1^2 x e^{\lambda_1 x} = \lambda_1^2 x e^{\lambda_1 x} + 2\lambda_1 e^{\lambda_1 x}.$$

Hence

$$\begin{aligned} y'' + ry' + sy &= \lambda_1^2 x e^{\lambda_1 x} + 2\lambda_1 e^{\lambda_1 x} + re^{\lambda_1 x} + r\lambda_1 x e^{\lambda_1 x} + sxe^{\lambda_1 x} \\ &= \underbrace{(\lambda_1^2 + r\lambda_1 + s)}_{=0} xe^{\lambda_1 x} + \underbrace{(2\lambda_1 + r)}_{=0} e^{\lambda_1 x} \\ &= 0. \end{aligned}$$

That is, $xe^{\lambda_1 x}$ is a solution to (5.10). Also, $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$ are independent. Therefore, in this case the general solution of (5.10) is

$$g(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}.$$

Example 5.18. Consider the differential equation

$$y'' + 2y' + y = 0.$$

The auxiliary equation is

$$\lambda^2 + 2\lambda + 1 = 0 \iff (\lambda + 1)^2 = 0 \implies \lambda_1 = -1,$$

where λ_1 is the only root. But $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$ are two independent solutions. So the general solution is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} = e^{-x}(C_1 + C_2 x).$$

(c) $\Delta < 0$, there is no real solution to (5.12) (see Fig. 5.1(c)). But it is still possible to find two independent solutions, these will be complex roots given by

$$\lambda = \frac{-r \pm \sqrt{\Delta}}{2} = \frac{-r \pm \sqrt{(-1)(-\Delta)}}{2} = -\frac{r}{2} \pm \frac{1}{2}\sqrt{-1}\sqrt{-\Delta} = \alpha + i\beta,$$

where $-\Delta > 0$ and

$$i = \sqrt{-1}, \quad \alpha = -\frac{r}{2}, \quad \beta = \frac{1}{2}\sqrt{-\Delta} = \frac{1}{2}\sqrt{4s - r^2}.$$

In this case, the solutions are $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$, so we have the general solution

$$g(x) = \tilde{C}_1 e^{(\alpha+i\beta)x} + \tilde{C}_2 e^{(\alpha-i\beta)x} = e^{\alpha x}\left(\tilde{C}_1 e^{i\beta x} + \tilde{C}_2 e^{-i\beta x}\right).$$

But recall Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, which means we can write the general solution as

$$g(x) = e^{\alpha x}\left((\tilde{C}_1 + \tilde{C}_2)\cos\beta x + (\tilde{C}_1 - \tilde{C}_2)\sin\beta x\right) = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right),$$

where $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are two independent solutions.

Let us verify

$$y = e^{\alpha x}\cos\beta x$$

is a solution of (5.10). We have

$$y' = \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x,$$

$$\begin{aligned} y'' &= \alpha^2 e^{\alpha x}\cos\beta x - \alpha\beta e^{\alpha x}\sin\beta x - \alpha\beta e^{\alpha x}\sin\beta x - \beta^2 e^{\alpha x}\cos\beta x \\ &= (\alpha^2 - \beta^2)e^{\alpha x}\cos\beta x - 2\alpha\beta e^{\alpha x}\sin\beta x. \end{aligned}$$

Plugging these into equation (5.10), we have

$$\begin{aligned} y'' + ry' + sy = (\alpha^2 - \beta^2)&e^{\alpha x}\cos\beta x - 2\alpha\beta e^{\alpha x}\sin\beta x + \\ &+ r\alpha e^{\alpha x}\cos\beta x - r\beta e^{\alpha x}\sin\beta x + s e^{\alpha x}\cos\beta x, \end{aligned}$$

which can be written as

$$y'' + ry' + sy = \underbrace{(\alpha^2 - \beta^2 + \alpha r + s)}_{\frac{r^2}{4} - \frac{1}{4}(4s - r^2) - \frac{r^2}{2} + s = 0} e^{\alpha x}\cos\beta x - \underbrace{(2\alpha + r)}_{=0}\beta e^{\alpha x}\sin\beta x = 0.$$

Similarly we can show $y = e^{\alpha x}\sin\beta x$ also satisfies (5.10), thus $g(x)$ above is a general solution to (5.10).

Example 5.19. Consider the differential equation

$$y'' - 6y' + 13y = 0.$$

The auxiliary equation is

$$\lambda^2 - 6\lambda + 13 = 0,$$

which has

$$r = -6, \quad s = 13, \quad \Delta = r^2 - 4s = 36 - 52 = -16 < 0,$$

i.e. it has complex roots. So we have

$$\alpha = -\frac{r}{2} = 3, \quad \beta = \frac{1}{2}\sqrt{-(-16)} = 2,$$

therefore $e^{3x}\cos 2x$ and $e^{3x}\sin 2x$ are two independent solutions, so

$$y = e^{3x}(C_1 \cos 2x + C_2 \sin 2x),$$

is the general solution.[5]

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[5] End Lecture 25.

Finding a particular solution to $y'' + ry' + sy = p(x)$

It depends on the function $p(x)$. We only consider three kinds of $p(x)$:

1. Polynomials,
2. trigonometric functions,
3. the exponential functions:
    i. $e^{\mu x}$, μ is not a root of of the homogeneous equation $y'' + ry' + sy = 0$,
    ii. $e^{\mu x}$, μ is a root of the homogenous equation.

Example 5.20. Find the general solution to the differential equation

$$y'' + 2y' + y = x^2.$$

Recall, the general solution takes the form $y = f(x) + g(x)$. First we find the general solution to the homogeneous equation

$$y'' + 2y' + y = 0,$$

i.e. we seek the complimentary function (C.F.) $y = g(x)$. The auxiliary equation is

$$\lambda^2 + 2\lambda + 1 \quad \Longleftrightarrow \quad (\lambda + 1)^2 = 0 \quad \Longrightarrow \quad \lambda_1 = -1,$$

i.e. it has only one root. So the C.F. is

$$g = C_1 e^{-x} + C_2 x e^{-x} = (C_1 + C_2 x) e^{-x}.$$

Second, we find the particular integral (P.I.), we try

$$f = ax^2 + bx + x, \quad \text{since } p(x) = x^2.$$

so we have

$$f' = 2ax + b, \quad f'' = 2a.$$

Substituting $y = f(x)$ into the differential equation gives

$$\begin{aligned} f'' + 2f' + f &= 2a + 2(2ax + b) + ax^2 + bx + c \\ &= ax^2 + (4a + b)x + 2a + 2b + c \\ &\equiv x^2. \end{aligned}$$

Comparing coefficients between the LHS and the RHS we have

$$\left.\begin{array}{l} a = 1 \\ 4a + b = 0 \\ 2a + 2b + c = 0 \end{array}\right\} \quad \Longrightarrow \quad \left.\begin{array}{l} a = 1 \\ b = -4 \\ c = 6 \end{array}\right\} \quad \Longrightarrow \quad f(x) = x^2 - 4x + 6,$$

Finally, we can write the general solution as

$$y(x) = x^2 - 4x + 6 + (C_1 + C_2 x) e^{-x}.$$

Example 5.21. Solve the following initial-value problem:

$$y'' - 2y' + y = \sin x, \quad y(0) = -2, \quad y'(0) = 2.$$

Note, we have two conditions here because second order differential equations have two constants of integration to be found. The auxiliary equation for this problem is

$$\lambda^2 - 2\lambda + 1 = 0 \quad \iff \quad (\lambda - 1)^2 = 0 \implies \lambda = 1,$$

i.e. we have a repeated root and hence, the C.F. is

$$g(x) = (C_1 + C_2 x)e^x.$$

To find the P.I. we try

$$f = a \sin x + b \cos x,$$

since $p(x) = \sin x$. Then we have

$$f' = a \cos x - b \sin x, \quad f'' = -a \sin x - b \cos x.$$

Substituting into the differential equation we have

$$\begin{aligned} f'' - 2f' + f &= -a \sin x - b \cos x - 2a \cos x + 2b \sin x + a \sin x + b \cos x \\ &= (-a + 2b + a) \sin x + (-b - 2a + b) \cos x \\ &= 2b \sin x - 2a \cos x \\ &\equiv \sin x. \end{aligned}$$

Comparing coefficients, we have

$$a = 0, \quad b = \frac{1}{2} \quad \implies \quad f = \frac{1}{2} \cos x.$$

Therefore the general solution to the initial-value problem is

$$y(x) = \frac{1}{2} \cos x + (C_1 + C_2 x)e^x.$$

In order to find the unknown constants C_1 and C_2 using the initial conditions, we need to find $y'(x)$, so we differentiate the above to give

$$y'(x) = -\frac{1}{2} \sin x + C_2 e^x + (C_1 + C_2 x)e^x = -\frac{1}{2} \sin x + (C_1 + C_2 + C_2 x)e^x.$$

Put $x = 0$, so

$$y(0) = \frac{1}{2} \cos 0 + e^0 (C_1 + C_2 \cdot 0) = \frac{1}{2} + C_1 = -2,$$

$$y'(0) = -\frac{1}{2} \sin 0 + e^0 (C_1 + C_2 + C_2 \cdot 0) = C_1 + C_2 = 2,$$

thus, we have the constants

$$C_1 = -\frac{5}{2}, \quad C_2 = 2 - C_1 = 2 + 52 = \frac{9}{2}.$$

Finally, the solution to the initial value problem is

$$y(x) = \frac{1}{2} \cos x + \frac{1}{2} e^x (9x - 5).$$

Example 5.22. Find the general solution of the following differential equation,

$$y'' + 4y' + 3y = 5e^{4x}.$$

The auxiliary equation is

$$\lambda^2 + 4\lambda + 3 = 0 \qquad \Longleftrightarrow \qquad (\lambda + 1)(\lambda + 3) = 0.$$

Hence, this has two distinct real roots, namely $\lambda_1 = -1$, $\lambda_2 = -3$. So the C.F. (from the homogeneous equation) is given by

$$g(x) = C_1 e^{-x} + C_2 e^{-3x}.$$

To find the P.I. we try

$$f = ae^{4x},$$

since $p(x) = 5e^4x$. Differentiating, we have

$$f' = 4a^{4x}, \quad f'' = 16a^{4x}.$$

Substituting into the differential equation we see that

$$f'' + 4f' + 3f = 16ae^{4x} + 16ae^{4x} + 3ae^{4x} = 35ae^{4x} \equiv 5e^{4x}.$$

Therefore we have $a = 1/7$ and so the P.I. is

$$f = \frac{1}{7}a^{4x}.$$

Finally, the general solution is

$$y(x) = f(x) + g(x) = \frac{1}{7}e^{4x} + C_1 e^{-x} + C_2 e^{-3x}.$$

Example 5.23. Solve the initial-value problem given by

$$y'' + 4y' + 3y = e^{-x}, \quad y(0) = 0, \quad y'(0) = 0.$$

We know from example 5.22 that the C.F. to the homogeneous equation $y'' + 4y' + 3y = 0$ is

$$g = C_1 e^{-x} + C_2 e^{-3x},$$

i.e. $\lambda_1 = -1$ and $\lambda_2 = -3$.

Now let us find the P.I., if we try $f = ae^{-x}$, we know that it wouldn't work since ae^{-x} is actually a solution to the homogeneous equation $y'' + 4y' + 3y = 0$. Therefore, we try

$$f = axe^{-x},$$

thus

$$f' = ae^{-x} - axe^{-x}, \quad \text{and} \quad f'' = -2ae^{-x} + axe^{-x}.$$

Substituting into the differential equation we have

$$\begin{aligned} f'' + 4f' + 3f &= -2ae^{-x} + axe^{-x} + 4ae^{-x} - 4axe^{-x} + 3axe^{-x} \\ &= 2ae^{-x} \\ &\equiv e^{-x}. \end{aligned}$$

Therefore we must have $a = 1/2$ and so the general solution to the differential equations is

$$y(x) = \frac{1}{2}xe^{-x} + C_1e^{-x} + C_2e^{-3x}.$$

Differentiating the general solution we have

$$y'(x) = \frac{1}{2}e^{-x} - \frac{1}{2}xe^{-x} - C_1e^{-x} - 3C_2e^{-3x}.$$

Putting $x = 0$, we have (from the initial conditions)

$$\left.\begin{array}{l} y(0) = C_1 + C_2 = 0 \\ y'(0) = \frac{1}{2} - C_1 - 3C_2 = 0 \end{array}\right\} \quad \Longrightarrow \quad \begin{array}{l} C_1 = -\frac{1}{4} \\ C_2 = \frac{1}{4} \end{array},$$

thus

$$y(x) = \frac{1}{2}xe^{-x} - \frac{1}{4}e^{-x} + \frac{1}{4}e^{-3x},$$

is the solution to the initial-value problem.[6]

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[6] End Lecture 26.

Example 5.24. Find the general solution to the equation

$$y'' + 2y' + y = 2e^{-x}.$$

The general solution takes the form $y(x) = f(x) + g(x)$.

Find the C.F.: The auxiliary equation for the above differential equation is

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Longleftrightarrow \quad (\lambda+1)^2 = 0 \quad \Longrightarrow \quad \lambda_1 = -1,$$

i.e. we have a repeated root so

$$g(x) = C_1 e^{-x} + C_2 x e^{-x}.$$

Here e^{-x} and xe^{-x} are two independent solutions to the homogeneous equation

$$y'' + 2y' + y = 0.$$

Find a P.I.: We have to try

$$f = ax^2 e^{-x},$$

since e^{-x} and xe^{-x} can't be the solution to the original differential equation as they satisfy the homogeneous equation. So we work out the derivatives

$$f' = 2axe^{-x} - ax^2 e^{-x},$$

$$f'' = 2ae^{-x} - 2axe^{-x} - 2axe^{-x} + ax^2 e^{-x} = axe^{-x} - 4axe^{-x} + 2ae^{-x}.$$

Substituting $y = f(x)$ into the differential equation, we have

$$\begin{aligned} f'' + 2f' + f &= ax^2e^{-x} - 4axe^{-x} + 2ae^{-x} + 4axe^{-x} - 2ax^2e^{-x} + ax^2e^{-x} \\ &= 2ae^{-x} \\ &\equiv 2e^{-x}, \end{aligned}$$

therefore we have $a = 1$. So finally, we have the general solution

$$y(x) = (C_1 + C_2 x + x^2)e^{-x}.$$

5.2 Simple Harmonic Motion (SHM)

SHM is essentially standard trigonometric oscillation at a single frequency, for example a pendulum.

An ideal pendulum consists of a weightless rod of length l attached at one end to a frictionless hinge and supporting a body of mass m at the other end. We describe the motion in terms of angle θ, made by the rod and the vertical.

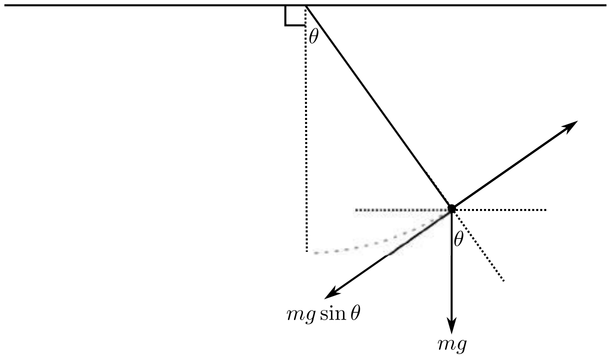

Figure 5.2: *Sketch of a pendulum of length l with a mass m, displaying the forces acting on the mass resolved in the tangential direction relative to the motion.*

Using Newton's second law of motion $F = ma$, we have the differential equation

$$-mg\sin\theta = ml\ddot{\theta},$$

which describes the motion of the mass m, where the RHS is the tangential acceleration and the LHS is the tangential component of gravitation force.

NOTATION: $\dot{\theta} = d\theta/dt$ and $\ddot{\theta} = d^2\theta/dt^2$.

We re-write the equation as

$$\ddot{\theta} + \omega^2\sin\theta = 0, \quad \omega^2 = \frac{g}{l}.$$

This is a nonlinear equation, and we can not solve it analytically.

Approximation: if θ is small, then $\sin\theta \approx \theta$, and in this situation we have an approximate equation given by

$$\ddot{\theta} + \omega^2\theta = 0.$$

We solve the equation for $\theta(t)$. Here we have $r = 0$, $s = \omega^2$ and $\Delta = -4\omega^2 < 0$. The auxiliary equation is

$$\lambda^2 + \omega^2 = 0 \quad \Longleftrightarrow \quad \lambda^2 = -\omega^2 \quad \Longrightarrow \quad \lambda_1 = i\omega, \quad \lambda_2 = -i\omega,$$

where $i = \sqrt{-1}$. So

$$\alpha = -\frac{r}{2} = 0 \quad \text{and} \quad \beta = \frac{1}{2}\sqrt{-1} = \omega,$$

therefore we have $e^{\alpha t}\cos\beta t = \cos\omega t$ and $e^{\alpha t}\sin\beta t = \sin\omega t$. Hence, the general solution is

$$\theta(t) = A\cos\omega t + B\sin\omega t.$$

Differentiating we have

$$\begin{aligned}\dot{\theta}(t) &= -A\omega\sin\omega t + B\omega\cos\omega t\\ \ddot{\theta}(t) &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t.\end{aligned}$$

We can verify $\theta(t)$ satisfies the original differential equation as

$$\ddot{\theta}(t) = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t = -\omega^2 \underbrace{(A \cos \omega t + B \sin \omega t.)}_{=\theta(t)} = -\omega^2 \theta(t).$$

The solution $\theta(t)$ can be written as

$$\begin{aligned} \theta(t) &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}} \cos \omega t + \frac{B}{\sqrt{A^2+B^2}} \sin \omega t\right) \\ &= \sqrt{A^2+B^2}(\sin \phi \cos \omega t + \cos \phi \sin \omega t) \\ &= R \sin(\omega t + \phi). \end{aligned}$$

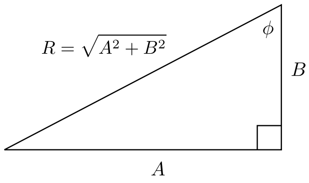

Figure 5.3: *Using Pythagarus' theorem to write the constants A and B in terms of the phase angle ϕ.*

R - amplitude of the motion.

ϕ - phase angle, i.e. the amount of shift.

$\omega = \sqrt{g/l}$ - the natural frequency, i.e. the number of complete oscillations per unit time.

$T = 2\pi/\omega$ - period, the time taken for a complete cycle (two complete swings). T depends on the length of the pendulum, but doesn't depend on the mass and initial conditions.

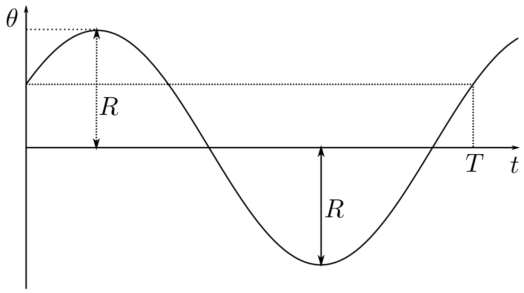

Figure 5.4: *Graph showing the change in θ over time t, displaying oscillations of period T.*

If the pendulum is initially at rest, i.e. $\theta(0) = 0$, $\dot{\theta}(0) = 0$, then

$$\theta(0) = A = 0, \quad \dot{\theta}(0) = B\omega = 0 \quad \Longrightarrow \quad B = 0 \quad \Longrightarrow \quad \theta(t) = 0,$$

i.e. the pendulum will remain at test for all time t.

If the pendulum is displaced by an angle θ_0 and released, then $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, so

$$\theta(0) = A = \theta_0, \quad \dot{\theta}(0) = B\omega = 0 \quad \Longrightarrow \quad B = 0,$$

therefore

$$\theta(t) = \theta_0 \cos \omega t \quad \Longrightarrow \quad |\theta(t)| \leq \theta_0.$$

That is, if the displaced angle θ_0 at initial time is small, then the small angle approximation makes sense.

The solution tells us the oscillation, once started, goes forever. But in reality, the frictional force and air resistance would eventually bring the pendulum to a rest.[7]

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[7] End Lecture 27.

5.3 Solving initial-value problems numerically: Euler's method

Most differential equations can not be solved analytically, so we try to solve them numerically.

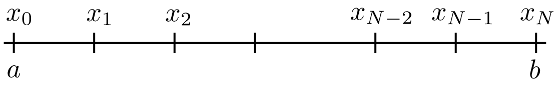
Suppose we have an initial-value problem:

$$\frac{dy}{dx} = f(x, y), \quad y(a) = y_0.$$

We want to find the solution $y(x)$ numerically on the interval $[a, b]$.

First we divide $[a, b]$ into N subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_k < \cdots < x_N = b.$$



If the subintervals have equal length, say h, then

$$x_k = a + kh, \quad h = \frac{b-a}{N} \quad \text{(step size)}, \quad k = 0, 1, \ldots, N.$$

Assume that $y(x)$ is the solution we want, then

$$\frac{d}{dx} y(x) = f(x, y(x)), \quad y(a) = y_0.$$

Integrating both sides on the subinterval $[x_k, x_{k+1}]$, we have

$$\int_{x_k}^{x_{k+1}} y'(x)\, dx = \int_{x_k}^{x_{k+1}} f(x, y(x))\, dx.$$

Considering the LHS, we have

$$\text{LHS} = y(x)\big|_{x_k}^{x_{k+1}} = y(x_{k+1}) - y(x_k),$$

thus

$$y(x_{k+1}) - y(x_k) = \int_{x_k}^{x_{k+1}} f(x, y(x))\, dx.$$

Let $g(x) = f(x, y(x))$, then

$$\text{RHS} = \int_{x_k}^{x_{k+1}} g(x)\, dx,$$

representing the area under the curve $y = g(x)$, between x_k and x_{k+1}.

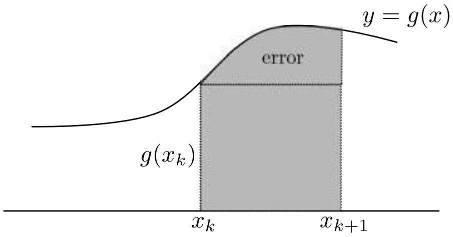


Figure 5.5: *The integral which represents the area under the curve $y = g(x)$ is approximated using rectangles. Error depends on the width of the rectangles, i.e. the number of sub-intervals.*

If we use the area of the rectangle

$$(x_{k+1} - x_k)g(x_k) = hg(x_k) = hf(x_k, y(x_k)),$$

to approximate the area, we have

$$y(x_{k+1}) - y(x_k) \approx hf(x_k, y(x_k)).$$

$k = 0$: $y(x_1) - y_0 \approx h(f(x_0, y_0) \implies y(x_1) \approx y_0 + hf(x_0, y_1) \triangleq y_1$, an approximation to $y(x_1)$.

$k = 1$: $y(x_2) - y(x_1) \approx h(f(x_1, y(x_1)) \implies y(x_2) \approx y(x_1) + hf(x_1, y(x_1)) \approx y_1 + hf(x_1, y_1) \triangleq y_2$, an approximation to $y(x_2)$.

$\vdots$

In general, we have

$$y_{k+1} = y_k + hf(x_k, y_k), \quad k = 0, 1, \dots N - 1,$$

where y_k is an approximation to $y(x_k)$. This is a *difference equation* and we can solve it iteratively. This method is based on the above formula, and is called *Euler's method.*

Example 5.25. Estimate $y(1)$, where $y(x)$ satisfies the initial-value problem:

$$\frac{dy}{dx} = y, \quad y(0) = 1.$$

We know the exact solution is

$$y(x) = e^x, \quad \implies \quad y(1) = e \approx 2.71828.$$

Now we apply Euler's method to the problem. We have

$$f(x, y) = y.$$

First, we take $N = 5$, then $h = (1 - 0)/5 = 0.2$.

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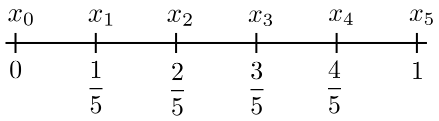

$$
\begin{aligned}
y_0 &= y(0) = 1\\
y_1 &= y_0 + hf(x_0, y_0) = 1 + 0.2 \times 1 = 1.2\\
y_2 &= y_1 + hf(x_1, y_1) = 1.2 + 0.2 \times 1.2 = (1.2)^2\\
y_3 &= y_2 + hf(x_2, y_2) = y_2 + hy_2 = y_2(1+h) = (1.2)^2 \times 1.2 = (1.2)^3\\
y_4 &= (1.2)^4\\
y_5 &= (1.2)^5 \approx 2.48832.
\end{aligned}
$$

For $N = 5$, we have

$$\text{error} = e - y_5 = 2.71828 - 2.48832 = 0.22996.$$

Now, we double the number of subintervals: $N = 10$, $h = 0.1$ then we need 10 steps to reach $x_{10} = 1$.

$$y_{10} = (1.1)^{10} \approx 2.59374,$$

then we have

$$\text{error} = 2.71828 - 2.59374 = 0.12454.$$

For $N = 20$, $h = 0.05$ and so

$$y_{20} = (1.05)^{20} \approx 2.65330, \quad \text{error} = 0.0650.$$

For $N = 40$, $h = 0.025$ and so

$$y_{20} = (1.025)^{40} \approx 2.68506, \quad \text{error} = 0.0332.$$

Euler's method is first order, i.e. the error behaves like $O(h)$.

In general, if $h = 1/N$, then

$$
\begin{aligned}
y_1 &= y_0 + hy_0 = (1+h)y_0 = 1 + h = 1 + \frac{1}{N}\\
y_2 &= y_1 + hy_1 = (1+h)y_1 = \left(1 + \frac{1}{N}\right)^2\\
&\vdots\\
y_N &= \left(1 + \frac{1}{N}\right)^N.
\end{aligned}
$$

Thus

$$y(1) \approx \left(1 + \frac{1}{N}\right)^N.$$

Actually,

$$\lim_{N\to\infty} \left\{\left(1 + \frac{1}{N}\right)^N\right\} = e.$$

The source of the errors when approximating the function come from

1. discretisation error,
2. round-off error.[8]

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[8] End Lecture 28. End of course.